Laplace Transforms

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1 LAPLACE TRANSFORM

A relation of the form $F(s) = \int_a^b k(s,t) f(t)\,\mathrm{d}t$ which transforms a given function $f(t)$ into another function $F(s)$, is called an integral transform. Here $K(s,t)$ is called the kernal of the transform and $F(s)$ the transform of $f(t)$. The most common integral transforms are

$$\text{(i)} \quad a = 0, \quad b = \infty, \quad K(s,t) = \mathrm{e}^{-st}, \quad \text{(Laplaces)}$$

$$\text{(ii)} \quad a = -\infty, \quad b = \infty, \quad K(s,t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{ist} \quad \text{(Fourier)}$$

The idea behind any transform is that given problem can be solved more easily in the transformed domain. Laplace transform reduces the problem of solving a differential equation to an algebraic problem. It is widely used in problems where the

Definition 1.1

Let $f(t)$ be a function defined for all $t \geq 0$. The Laplace transform of $f(t)$ is defined as

$$\mathscr{L}[f(t)] = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t = F(s), \tag{1}$$

provided the integral exists. Here s is a parameter real or complex. The function $f(t)$ whose transform is $F(s)$ is said to be the inverse transform of $F(s)$ and is denoted by $\mathscr{L}^{-1}[F(s)]$. Thus if $\mathscr{L}[f(t)] = F(s)$, then $f(t) = \mathscr{L}^{-1}[F(s)]$.

EXISTENCE OF LAPLACE TRANSFORM

A function $f(t)$ is said to be of *exponential order* or satisfies *growth restriction* if there exist constants M and a such that $|f(t)| \leq M\mathrm{e}^{at}$ for all positive t.

Theorem 1.1 (Existence Theorem for Laplace Transforms)

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and is of exponential order for all for all $t \geq 0$ and some constants M and a, then the Laplace transform of $f(t)$ exists for all $s > a$.

Proof. Since $f(t)$ is piecewise continuous, $\mathrm{e}^{-st} f(t)$ is integrable over any finite interval on the

t-axis. Also $f(t)$ is of exponential order so that for $s > a$ we get,

$$\begin{aligned}
|\mathscr{L}[f(t)]| &= \left| \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \right| \\
&\leq \int_0^\infty |\mathrm{e}^{-st} f(t)|\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-st} |f(t)|\,\mathrm{d}t \\
&\leq \int_0^\infty \mathrm{e}^{-st} M \mathrm{e}^{at}\,\mathrm{d}t \\
&= M \lim_{k\to\infty} \int_0^k \mathrm{e}^{-(s-a)t}\,\mathrm{d}t \\
&= M \lim_{k\to\infty} \left[\frac{\mathrm{e}^{-(s-a)t}}{-(s-a)} \right]_0^k \\
&= M \lim_{k\to\infty} \left[\frac{\mathrm{e}^{-(s-a)k}}{-(s-a)} - \frac{1}{-(s-a)} \right] \\
&= \frac{M}{s-a}.
\end{aligned}$$

Hence $\mathscr{L}[f(t)]$ exists for $s > a$. ■

► Note that

$$\int_a^\infty f(x)\,dx = \lim_{k\to\infty} \int_a^k f(x)\,dx$$

provided, the limit on the right side exist.

LINEARITY OF THE LAPLACE TRANSFORM

The Laplace transform is a linear operation. i.e. for any function $f(t)$ and $g(t)$ whose Laplace transform exist and any constants a and b,

$$\mathscr{L}[af(t) + bg(t)] = a\mathscr{L}[f(t)] + b\mathscr{L}[g(t)].$$

Proof. By definition,

$$\begin{aligned}
\mathscr{L}[af(t) + bg(t)] &= \int_0^\infty \mathrm{e}^{-st}[af(t) + bg(t)]\,\mathrm{d}t \\
&= a \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t + b \int_0^\infty \mathrm{e}^{-st} g(t)\,\mathrm{d}t \\
&= a\mathscr{L}[f(t)] + b\mathscr{L}[g(t)].
\end{aligned}$$

■

2 LAPLACE TRANSFORM OF SOME ELEMENTARY FUNCTIONS

▶ Note that $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to\infty} e^{-ax} = 0$ for any $a > 0$.

(i) $\mathscr{L}(1) = \frac{1}{s}, \quad s > 0.$

Proof. By definition,

$$\begin{aligned}
\mathscr{L}(1) &= \int_0^\infty e^{-st} 1 \, dt \\
&= \lim_{k\to\infty} \int_0^k e^{-st} 1 \, dt \\
&= \lim_{k\to\infty} \left(\frac{e^{-st}}{-s} \right)_0^k \\
&= \lim_{k\to\infty} \left(\frac{e^{-sk}}{-s} + \frac{1}{s} \right) \\
&= \left(0 + \frac{1}{s} \right) \\
&= \frac{1}{s}, \quad s > 0.
\end{aligned}$$

■

(ii) $\mathscr{L}(e^{at}) = \frac{1}{s-a}, \quad s > a.$

Proof.

$$\begin{aligned}
\mathscr{L}[e^{at}] &= \int_0^\infty e^{-st} e^{at} \, dt \\
&= \int_0^\infty e^{-(s-a)t} \, dt \\
&= \lim_{k\to\infty} \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^k \\
&= \lim_{k\to\infty} \left[\frac{e^{-(s-a)k}}{-(s-a)} + \frac{1}{s-a} \right] \\
&= \frac{1}{s-a} \text{ if } s > a.
\end{aligned}$$

■

(iii) $\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$ where n is a positive integer.

Proof. We prove this formula by induction. For $n = 0$, $t^n = t^0 = 1$ and $\mathscr{L}(1) = \frac{1}{s}$. Thus the result is true for $n = 0$. We now make the induction hypothesis that it holds for any positive integer n. i.e. $\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$.

Now

$$\begin{aligned}
\mathscr{L}(t^{n+1}) &= \int_0^\infty \mathrm{e}^{-st} t^{n+1}\,\mathrm{d}t \\
&= \int_0^\infty t^{n+1}\,\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \lim_{k\to\infty} \int_0^k t^{n+1}\,\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \lim_{k\to\infty} \left[t^{n+1} \frac{\mathrm{e}^{-st}}{-s}\bigg|_0^k - \int_0^k (n+1)t^n \frac{\mathrm{e}^{-st}}{-s}\,\mathrm{d}t \right] \\
&= \lim_{k\to\infty} \left[k^{n+1} \frac{\mathrm{e}^{-sk}}{-s} - 0 - \int_0^k (n+1)t^n \frac{\mathrm{e}^{-st}}{-s}\,\mathrm{d}t \right] \\
&= [0-0] + \frac{n+1}{s} \int_0^\infty \mathrm{e}^{-st} t^n\,\mathrm{d}t \\
&= \frac{n+1}{s} \mathscr{L}(t^n) \\
&= \frac{n+1}{s} \frac{n!}{s^{n+1}} \\
&= \frac{(n+1)!}{s^{n+2}}. \qquad \blacksquare
\end{aligned}$$

Thus the result is true for $n+1$. This proves formula (iii).

(iv) $\mathscr{L}[\sin at] = \dfrac{a}{s^2+a^2}, \quad s>0$

Proof. We have $\mathscr{L}[\sin at] = \int_0^\infty \mathrm{e}^{-st} \sin at\,\mathrm{d}t = \lim_{k\to\infty} \int_0^k \mathrm{e}^{-st} \sin at\,\mathrm{d}t$. We first evaluate $\int \mathrm{e}^{-st} \sin at\,\mathrm{d}t$. Let $I = \int \mathrm{e}^{-st} \sin at\,\mathrm{d}t$, then

$$\begin{aligned}
I &= \mathrm{e}^{-st} \frac{-\cos at}{a} - \int (-s)\mathrm{e}^{-st} \frac{-\cos at}{a}\,dt \\
&= \frac{-\mathrm{e}^{-st}\cos at}{a} - \frac{s}{a} \int \mathrm{e}^{-st} \cos at\,dt \\
&= \frac{-\mathrm{e}^{-st}\cos at}{a} - \frac{s}{a} \left[\mathrm{e}^{-st} \frac{\sin at}{a} - \int (-s)\mathrm{e}^{-st} \frac{\sin at}{a}\,dt \right] \\
&= \frac{-\mathrm{e}^{-st}\cos at}{a} - \frac{s}{a^2}\mathrm{e}^{-st} \sin at - \frac{s^2}{a^2} \int \mathrm{e}^{-st} \sin at\,dt \\
&= \frac{-\mathrm{e}^{-st}\cos at}{a} - \frac{s}{a^2}\mathrm{e}^{-st} \sin at - \frac{s^2}{a^2} I \\
\Rightarrow I + \frac{s^2}{a^2} I &= \frac{-\mathrm{e}^{-st}\cos at}{a} - \frac{s}{a^2}\mathrm{e}^{-st} \sin at \\
\Rightarrow \frac{(s^2+a^2)I}{a^2} &= \frac{-\mathrm{e}^{-st}\cos at}{a} - \frac{s}{a^2}\mathrm{e}^{-st} \sin at \\
\Rightarrow (s^2+a^2)I &= -a\mathrm{e}^{-st} \cos at - s\mathrm{e}^{-st} \sin at \\
&= \mathrm{e}^{-st}(-a\cos at - s\sin at) \\
\Rightarrow I &= \frac{\mathrm{e}^{-st}}{s^2+a^2}(-a\cos at - s\sin at).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathscr{L}[\sin at] &= \lim_{k\to\infty}\int_0^k \mathrm{e}^{-st}\sin at\,\mathrm{d}t \\
&= \lim_{k\to\infty}\left[\frac{\mathrm{e}^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^k \\
&= \lim_{k\to\infty}\left[\frac{\mathrm{e}^{-sk}}{s^2+a^2}(-s\sin ak - a\cos ak) - \frac{1}{s^2+a^2}(0-a)\right] \\
&= 0 + \frac{a}{s^2+a^2} \\
&= \frac{a}{s^2+a^2},\ s>0.
\end{aligned}$$

■

▶ We can see that for $s>0$,

- ▶ $\lim_{x\to\infty}\mathrm{e}^{-sx}\sin ax = 0$
- ▶ $\lim_{x\to\infty}\mathrm{e}^{-sx}\cos ax = 0$

Proof. We have

$$\left|\mathrm{e}^{-sx}\sin ax\right| = \left|\frac{\sin ax}{\mathrm{e}^{sx}}\right| \le \left|\frac{1}{\mathrm{e}^{sx}}\right| = \mathrm{e}^{-sx}.$$

Since $\mathrm{e}^{-sx}\to 0$ as $x\to\infty$, we see that $\lim_{x\to\infty}\mathrm{e}^{-sx}\sin ax = 0$.

Similarly $\lim_{x\to\infty}\mathrm{e}^{-sx}\cos ax = 0$, since $|\cos ax|\le 1$. ■

(v) $\mathscr{L}(\cos at) = \dfrac{s}{s^2+a^2}, \quad s>0.$

Proof. $\mathscr{L}[\cos at] = \lim_{k\to\infty}\int_0^k \mathrm{e}^{-st}\cos at\,\mathrm{d}t$. As in the previous problem, we can see that $\int \mathrm{e}^{-st}\cos at\,\mathrm{d}t = \frac{\mathrm{e}^{-st}}{s^2+a^2}(-s\cos at + a\sin at)$. Using it,

$$\begin{aligned}
\mathscr{L}[\cos at] &= \int_0^\infty \mathrm{e}^{-st}\cos at\,\mathrm{d}t \\
&= \lim_{k\to\infty}\left[\frac{\mathrm{e}^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^k \\
&= \lim_{k\to\infty}\left[\frac{\mathrm{e}^{-sk}}{s^2+a^2}(-s\cos ak + a\sin ak) - \frac{1}{s^2+a^2}(-s+0)\right] \\
&= 0 + \frac{s}{s^2+a^2} \\
&= \frac{s}{s^2+a^2}.
\end{aligned}$$

■

(vi) $\mathscr{L}[\sinh at] = \dfrac{a}{s^2-a^2}, \quad s>|a|$

Proof. We have $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$

$$\begin{aligned}
\mathscr{L}[\sinh at] &= L\left[\frac{1}{2}(e^{+at} - e^{-at})\right] \\
&= \frac{1}{2}\{\mathscr{L}(e^{at}) - \mathscr{L}(e^{-at})\} \\
&= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s-(-a)}\right] \\
&= \frac{1}{2}\left[\frac{s+a-(s-a)}{(s-a)(s+a)}\right] \\
&= \frac{1}{2} \times \frac{2a}{s^2-a^2} \\
&= \frac{a}{s^2-a^2}.
\end{aligned}$$

■

(vii) $\mathscr{L}[\cosh at] = \dfrac{s}{s^2-a^2}, \quad s > |a|$

Proof.

$$\begin{aligned}
\mathscr{L}[\cosh at] &= L\left[\frac{e^{at} + e^{-at}}{2}\right] \\
&= \frac{1}{2}\{\mathscr{L}(e^{at}) + \mathscr{L}(e^{-at})\} \\
&= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] \\
&= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] \\
&= \frac{1}{2} \times \frac{2s}{s^2-a^2} \\
&= \frac{s}{s^2-a^2}.
\end{aligned}$$

■

Problem 2.1

Find the Laplace transform of $f(t) = \begin{cases} e^t, & 0 < t < 1 \\ 0 & \text{when } t > 1 \end{cases}$

Solution. We have

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^1 \mathrm{e}^{-st} f(t)\,\mathrm{d}t + \int_1^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^1 \mathrm{e}^{-st}\mathrm{e}^{t}\,\mathrm{d}t + \int_1^\infty \mathrm{e}^{-st} 0\,\mathrm{d}t \\
&= \int_0^1 \mathrm{e}^{-(s-1)t}\,\mathrm{d}t \\
&= \left[\frac{\mathrm{e}^{-(s-1)t}}{-(s-1)}\right]_0^1 \\
&= \frac{\mathrm{e}^{-(s-1)}}{-(s-1)} - \frac{1}{-(s-1)} \\
&= \frac{\mathrm{e}^{1-s}}{1-s} - \frac{1}{1-s} \\
&= \frac{e^{1-s}-1}{1-s}.
\end{aligned}$$

Problem 2.2

Find the Laplace transform of $f(t) = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & t > 2\pi \end{cases}$

Solution. We have

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \mathrm{e}^{-st} f(t)\,\mathrm{d}t + \int_{2\pi}^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \mathrm{e}^{-st} \cos t\,\mathrm{d}t + \int_{2\pi}^\infty \mathrm{e}^{-st} 0.\,\mathrm{d}t \\
&= \left\{\frac{\mathrm{e}^{-st}}{s^2+1}(-s\cos t + \sin t)\right\}_0^{2\pi} \\
&= \frac{\mathrm{e}^{-s2\pi}}{s^2+1}(-s) - \frac{1}{s^2+1}(-s) \\
&= (1-\mathrm{e}^{-2s\pi})\frac{s}{s^2+1}.
\end{aligned}$$

Problem 2.3

Find the Laplace transform of the following functions.

(i)	$\sin 3t \cos 2t$	(iii)	$\cos^3 2t$	(v)	$\sin 3t \sin 2t$
(ii)	$\sin^2 3t$	(iv)	$\sinh^3 2t$	(vi)	$\mathrm{e}^{-4t} - 6t^2 + 4\sin 2t$

Solution. 1. We have

$$\mathscr{L}[\sin 3t\cos 2t] = \frac{1}{2}\{\mathscr{L}(\sin 5t)+\mathscr{L}(\sin t)\}$$
$$= \frac{1}{2}\left\{\frac{5}{s^2+25}+\frac{1}{s^2+1}\right\}$$
$$= \frac{3(s^2+5)}{(s^2+1)(s^2+2s)}$$

2. we have

$$\mathscr{L}(\sin^2 3t) = L\left[\frac{1-\cos 6t}{2}\right] = \frac{1}{2}\{\mathscr{L}(1)-\mathscr{L}(\cos 6t)\}$$
$$= \frac{1}{2}\left\{\frac{1}{s}-\frac{s}{s^2+36}\right\} = \frac{18}{s(s^2+36)}$$

3. we have $\cos 3A = 4\cos^3 A - 3\cos A$ or $\cos^3 A = \frac{1}{4}\cos 3A + \frac{3}{4}\cos A$

$$\therefore \quad \cos^3 2t = \frac{1}{4}\cos 6t + \frac{3}{4}\cos 2t$$
$$\mathscr{L}(\cos^3 2t) = \frac{1}{4}\mathscr{L}(\cos 6t) + \frac{3}{4}\mathscr{L}(\cos 2t)$$
$$= \frac{1}{4}\times\frac{s}{s^2+36}+\frac{3}{4}\times\frac{s}{s^2+4}$$
$$= \frac{s(s^2+28)}{(s^2+4)(s^2+36)}$$

4. we have $\sinh 3A = 3\sinh A + 4\sinh^3 A$.

$$\text{i.e. } \sinh^3 A = \frac{1}{4}\sinh 3A - \frac{3}{4}\sinh A$$
$$\text{i.e. } \sinh^3 2t = \frac{1}{4}\sinh 6t - \frac{3}{4}\sinh 2t$$
$$\therefore \quad \mathscr{L}[\sinh^3 2t] = \frac{1}{4}\mathscr{L}[\sinh 6t] - \frac{3}{4}L[\sinh 2t]$$
$$= \frac{1}{4}\left[\frac{6}{s^2-36}\right]-\frac{3}{4}\left[\frac{2}{s^2-4}\right]$$
$$= \frac{3}{2}\times\frac{s^2-4-s^2+36}{(s^2-36)(s^2-4)}$$
$$= \frac{48}{(s^2-4)(s^2-36)}.$$

5. We have $\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$

$$\text{i.e. } \sin 3t\sin 2t = \frac{1}{2}[\cos t - \cos 5t]$$
$$\therefore \quad \mathscr{L}[\sin 3t\sin 2t] = \frac{1}{2}[\mathscr{L}(\cos t)-\mathscr{L}[(\cos 5t)]$$
$$= \frac{1}{2}\left[\frac{s}{s^2+1}-\frac{s}{s^2+25}\right]$$
$$= \frac{12s}{(s^2+1)(s^2+25)}$$

6. We have

$$\begin{aligned}\mathscr{L}[e^{-4t}-6t^2+4\sin 2t] &= \mathscr{L}(e^{-4t})-6\mathscr{L}(t^2)+4\mathscr{L}(\sin 2t)\\ &= \frac{1}{s-(-4)}-6\times\frac{2!}{s^{2+1}}+4\times\frac{2}{s^2+4}\\ &= \frac{1}{s+4}-\frac{12}{s^3}+\frac{8}{s^2+4}.\end{aligned}$$

■

Problem 2.4

Find the Laplace transforms of

1. $3e^{5t}+(t+2)^2+2\cos 3t$
2. $\sin(at+b)$

Solution. 1. $\mathscr{L}[3e^{5t}+(t+2)^2+2\cos 3t] = 3\mathscr{L}(e^{5t})+\mathscr{L}[(t+2)^2]+2\mathscr{L}[\cos 3t]$ (1)

But $\mathscr{L}[e^{5t}] = \frac{1}{s-5}$

$$\begin{aligned}\mathscr{L}((t+2)^2) &= \mathscr{L}[t^2+4t+4] = \mathscr{L}(t^2)+4\mathscr{L}(t)+\mathscr{L}(4)\\ &= \frac{2!}{s^3}+4\cdot\frac{1!}{s^2}+4\times\frac{1}{s} = \frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}.\end{aligned}$$

and $\mathscr{L}[\cos 3t) = \frac{s}{s^2+9}$

Substituting in (1)

$$\mathscr{L}[3e^{5t}+(t+2)^2+2\cos 3t] = \frac{3}{s-5}+\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}+\frac{2s}{s^2+9}$$

2. $\sin(at+b) = \sin at\cos b+\cos at\sin b$

$$\begin{aligned}\therefore\quad \mathscr{L}[\sin(at+b)] &= \cos b\mathscr{L}[\sin at]+\sin b\mathscr{L}(\cos at)\\ &= \cos b\times\frac{a}{s^2+a^2}+\sin b\times\frac{s}{s^2+a^2}\\ &= \frac{a\cos b+s\sin b}{s^2+a^2}\end{aligned}$$

■

EXERCISE

Find the Laplace transform of the following.

(1) $f(t) = \begin{cases} t/k, & 0 < t < k \\ 1, & t > k \end{cases}$

(2) $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

(3) $\cos t \cdot \cos 2t$

(4) $\sin 2t \cos 3t$

(5) $5e^{3t} + 3t^3 - 2\sin 3t + 3\cos 3t$

(6) $\sin^3 2t$

(7) $\cos(at + b)$

(8) $\cosh^3 2t$

(9) $\cos^2 3t + (t^2 + 1)^2$

ANSWERS

(1)	$\dfrac{1 - e^{-ks}}{ks^2}$	(2)	$\dfrac{1 + e^{-\pi s}}{s^2 + 1}$	(3)	$\dfrac{s(s^2 + 5)}{(s^2 + 1)(s^2 + 9)}$
(4)	$\dfrac{2(s^2 - 5)}{(s^2 + 1)(s^2 + 25)}$	(5)	$\dfrac{5}{s - 3} + \dfrac{18}{s^4} + \dfrac{3(s - 2)}{s^2 + 9}$		
(6)	$\dfrac{48}{(s^2 + 4)(s^2 + 36)}$	(7)	$\dfrac{s\cos b - a\sin b}{s^2 + a^2}$		
(8)	$\dfrac{s(s^2 - 28)}{(s^2 - 4)(s^2 - 36)}$	(9)	$\dfrac{s^2 + 18}{s(s^2 + 36)} + \dfrac{24}{s^5} + \dfrac{4}{s^3} + \dfrac{1}{s}$		

Theorem 2.1 (First shifting theorem, s-Shifting)

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at} f(t)$ has the transform $F(s - a)$ (where $s - a > k$). In formulas,

$$\mathscr{L}[e^{at} f(t)] = F(s - a)$$

or, if we take the inverse on both sides,

$$e^{at} f(t) = \mathscr{L}^{-1}[F(s - a)]$$

Proof. We have

$$F(s) = \mathscr{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt \tag{1}$$

$$\begin{aligned} \therefore \quad \mathscr{L}[e^{at} f(t)] &= \int_0^\infty e^{-st} e^{at} f(t)\,dt \\ &= \int_0^\infty e^{-(s-a)t} f(t)\,dt \\ &= F(s-a), \quad \text{by (1).} \end{aligned}$$

▶ Note that $\mathscr{L}[e^{-at} f(t)] = F(s+a)$.

Problem 2.5

Find the Laplace transform of the following:

(i) $e^{-3t} t^3$

(ii) $e^{-2t} \cos^2 t$

(iii) $\sinh at \cdot \sin at$

(iv) $e^{-2t} [\cos 4t + 3 \sin 4t]$

(v) $e^t \cosh 3t$

(vi) $5e^{2t} \sinh 2t$

(vii) $(t+1)^2 e^t$

Solution. (i) We have $\mathscr{L}[t^3] = \frac{3!}{s^4} = \frac{6}{s^4}$. By shifting property, we get

$$\mathscr{L}[e^{-3t} t^3] = \frac{6}{(s+3)^4} \quad \text{(Replacing } s \text{ by } s+3)$$

(ii) We have $\cos^2 t = \dfrac{1+\cos 2t}{2}$

$$\therefore \quad \mathscr{L}[\cos^2 t] = \frac{1}{2}\mathscr{L}[1+\cos 2t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right].$$

By shifting property,

$$\mathscr{L}[e^{-2t} \cos^2 t] = \frac{1}{2}\left[\frac{1}{s+2} + \frac{s+2}{(s+2)^2+4}\right] \quad \text{(Replacing } s \text{ by } s+2)$$

(iii) We have $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$ so that

$$\begin{aligned} \sinh at \cdot \sin at &= \left(\frac{e^{at} - e^{-at}}{2}\right) \sin at \\ &= \frac{1}{2}[e^{at} \sin at - e^{-at} \sin at] \\ \therefore \quad \mathscr{L}[\sinh at \sin at] &= \frac{1}{2}\left\{\mathscr{L}[e^{at} \sin at] - \mathscr{L}[e^{-at} \sin at]\right\} \qquad (1) \end{aligned}$$

But $\mathscr{L}[\sin at] = \dfrac{a}{s^2+a^2}$. By shifting property, $\mathscr{L}[e^{at} \sin at] = \dfrac{a}{(s-a)^2+a^2}$ and $\mathscr{L}[e^{-at} \sin at] = \dfrac{a}{(s+a)^2+a^2}$.

Substituting in (1),

$$\mathscr{L}[\sinh at \sin at] = \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right]$$

(iv) We have $\mathscr{L}[\cos 4t + 3\sin 4t] = \dfrac{s}{s^2+16} + 3 \times \dfrac{4}{s^2+16}$

$$= \frac{12+s}{s^2+16}.$$

By shifting property,

$$\mathscr{L}[e^{-2t}(\cos 4t + 3\sin 4t)] = \frac{12+(s+2)}{(s+2)^2+16} = \frac{s+14}{s^2+4s+20}$$

(v) we have $e^t \cosh 3t = e^t \left(\dfrac{e^{3t} + e^{-3t}}{2}\right) = \dfrac{e^{4t} + e^{-2t}}{2}$

$$\begin{aligned} \therefore \quad \mathscr{L}[e^t \cosh at] &= \frac{1}{2}[\mathscr{L}(e^{4t}) + \mathscr{L}(e^{-2t})] \\ &= \frac{1}{2} \times \left[\frac{1}{s-4} + \frac{1}{s+2}\right] \\ &= \frac{s-2}{s^2-2s-8} \end{aligned}$$

(vi) $5e^{2t} \sinh 2t = 5e^{2t}\left[\dfrac{e^{2t} - e^{-2t}}{2}\right] = \dfrac{5}{2}[e^{4t} - 1]$

$$\begin{aligned} \therefore \quad \mathscr{L}[5e^{2t} \sinh 2t] &= \frac{5}{2}[\mathscr{L}(e^{4t}) - \mathscr{L}(1)] = \frac{5}{2}\left[\frac{1}{s-4} - \frac{1}{s}\right] \\ &= \frac{10}{s(s-4)} \end{aligned}$$

or

We have $\mathscr{L}[5\sinh 2t] = 5 \times \dfrac{2}{s^2-4} = \dfrac{10}{s^2-4}$.

By shifting property,

$$\mathscr{L}[\mathrm{e}^{2t}5\sinh 2t] = \frac{10}{(s-2)^2-4} = \frac{10}{s^2-4s+4-4}$$
$$= \frac{10}{s(s-4)}$$

(vii) $\mathscr{L}[(t+1)^2] = \mathscr{L}(t^2+2t+1) = \mathscr{L}(t^2)+2\mathscr{L}(t)+\mathscr{L}(1)$

$$= \frac{2!}{s^2}+2\times\frac{1}{s^2}+\frac{1}{s} = \frac{2+2s+s^2}{s^3}$$

By shifting property, $$\mathscr{L}[\mathrm{e}^{t}(t+1)^2] = \frac{2+2(s-1)+(s-1)^2}{(s-1)^3}$$
$$= \frac{s^2+1}{(s-1)^3}.$$ ∎

Problem 2.6

If $\mathscr{L}[f(t)] = F(s)$, then show that $\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

Proof.

$$\mathscr{L}[f(at)] = \int_0^\infty \mathrm{e}^{-st}f(at)\,\mathrm{d}t \quad \text{put } at=u \quad a\,\mathrm{d}t=\mathrm{d}u$$
$$= \int_0^\infty \mathrm{e}^{-su/a}\cdot f(u)\cdot\frac{\mathrm{d}u}{a}$$
$$= \frac{1}{a}F(s/a).$$ ∎

3 TRANSFORMS OF DERIVATIVES AND INTEGRALS

The Laplace transform is a method of solving ODEs and initial value problems. The crucial idea is that operations of calculus on functions are replaced by operations of algebra on transforms. Roughly, differentiation of $f(t)$ will correspond to multiplication of $\mathscr{L}(f)$ by s and integration of $f(t)$ to division of $\mathscr{L}(f)$ by s. To solve ODEs, we must first consider the Laplace transform of derivatives. You have encountered such an idea in your study of logarithms. Under the application of the natural logarithm, a product of numbers becomes a sum of their logarithms, a division of numbers becomes their difference of logarithms.

Theorem 3.1

If $\mathscr{L}[f(t)] = F(s)$, then the transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathscr{L}[f'(t)] = sF(s) - f(0)$$
$$\mathscr{L}[f''(t)] = s^2F(s) - sf(0) - f'(0).$$

The first equation holds if $f(t)$ is continuous for all $t \geq 0$ and satisfies the growth restriction and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geq 0$. Similarly, second equation holds if f and f' are continuous for all $t \geq 0$ and satisfy the growth restriction and f'' is piecewise continuous on every finite interval on the semi-axis $t \geq 0$.

Proof. By definition,

$$\mathscr{L}[f'(t)] = \int_0^\infty \mathrm{e}^{-st} f'(t)\mathrm{d}t$$

Integrating by parts

$$\begin{aligned}
\mathscr{L}[f'(t)] &= \lim_{k\to\infty} \left[\mathrm{e}^{-st} f(t)\right]_0^k - \int_0^\infty \mathrm{e}^{-st}(-s)f(t)\mathrm{d}t \\
&= \lim_{k\to\infty} \left[\mathrm{e}^{-sk} f(k) - f(0)\right] - \int_0^\infty \mathrm{e}^{-st}(-s)f(t)\mathrm{d}t \\
&= 0 - f(0) + s\int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t \\
&= -f(0) + s\cdot\mathscr{L}[f(t)].
\end{aligned}$$

Thus

$$\mathscr{L}[f'(t)] = s\mathscr{L}[f(t)] - f(0).$$

Now using the formula for $\mathscr{L}[f'(t)]$, we get

$$\begin{aligned}
\mathscr{L}[f''(t)] &= -f'(0) + s\mathscr{L}[f'(t)] \\
&= -f'(0) + s[-f(0) + s\cdot\mathscr{L}[f(t)]] \\
&= -f'(0) - sf(0) + s^2\mathscr{L}[f(t)].
\end{aligned}$$

Hence

$$\mathscr{L}[f''(t)] = s^2F(s) - sf(0) - f'(0).$$ ■

▶ By applying the above result to higher order derivatives we get,

$$\mathscr{L}[f'''] = s^3\mathscr{L}[f] - s^2f(0) - sf'(0) - f''(0)$$

and so on. In general,

Theorem 3.2

Let $f, f', \dots, f^{(n-1)}$ be continuous for all $t \geq 0$ and satisfy the growth restriction. Furthermore, let f be piecewise continuous on every finite interval on the semi-axis $t \geq 0$. Then the transform of $f^{(n)}$ satisfies

$$\mathscr{L}[f^{(n)}] = s^n \mathscr{L}[f] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0). \qquad (2)$$

TRANSFORMS OF INTEGRALS

Theorem 3.3

Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geq 0$ and satisfies a growth restriction. Then, for $s > 0, s > k$, and $t > 0$,

$$\mathscr{L}\left\{\int_0^t f(u)\mathrm{d}u\right\} = \frac{F(s)}{s}, \quad \int_0^t f(u)\mathrm{d}u = \mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\} \qquad (3)$$

Proof. By definition,

$$\begin{aligned}
L\left\{\int_0^t f(u)\mathrm{d}u\right\} &= \int_0^\infty \mathrm{e}^{-st}\left(\int_0^t f(u)\mathrm{d}u\right)\mathrm{d}t \\
&= \int_0^\infty \left\{\int_0^t f(u)\mathrm{d}u\right\}\mathrm{e}^{-st}\mathrm{d}t \\
&= \lim_{k\to\infty}\int_0^k \left\{\int_0^t f(u)\mathrm{d}u\right\}\mathrm{e}^{-st}\mathrm{d}t \\
&= \lim_{k\to\infty}\left[\left(\int_0^t f(u)\mathrm{d}u\right)\frac{\mathrm{e}^{-st}}{-s}\bigg|_0^k - \int_0^k f(t)\frac{\mathrm{e}^{-st}}{-s}\mathrm{d}t\right] \\
&= \lim_{k\to\infty}\left[\left(\int_0^k f(u)\mathrm{d}u\right)\frac{\mathrm{e}^{-sk}}{-s} - \left(\int_0^0 f(u)\mathrm{d}u\right)\frac{1}{-s} - \int_0^k f(t)\frac{\mathrm{e}^{-st}}{-s}\mathrm{d}t\right] \\
&= \mathscr{L}\{f(t)\}\frac{0}{-s} - (0)\frac{1}{-s} - \int_0^\infty f(t)\frac{\mathrm{e}^{-st}}{-s}\mathrm{d}t \\
&= (0-0) + \frac{1}{s}\int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t \\
&= \frac{F(s)}{s}.
\end{aligned}$$

■

Problem 3.1

If $\mathscr{L}[f(t)] = \dfrac{1}{s(s^2+w^2)}$, find $f(t)$

Solution. Let

$$F(s) = \frac{1}{s^2+w^2}$$

Then

$$\frac{F(s)}{s} = \frac{1}{s(s^2+w^2)}$$

$$\begin{aligned}
\therefore f(t) &= \mathscr{L}^{-1}\left[\frac{1}{s(s^2+w^2)}\right] \\
&= \mathscr{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t \mathscr{L}^{-1}[F(s)]\mathrm{d}t \\
&= \int_0^t \mathscr{L}^{-1}\left[\frac{1}{s^2+w^2}\mathrm{d}t\right] \\
&= \int_0^t \frac{\sin wt}{w}\mathrm{d}t = \frac{1}{w}\left(\frac{-\cos wt}{w}\right)_0^t \\
&= -\frac{1}{w^2}[\cos wt - 1] = \frac{1 - \cos wt}{w^2}.
\end{aligned}$$

■

Problem 3.2

Find

$$\mathscr{L}^{-1}\left[\frac{1}{s^2(s^2+w^2)}\right]$$

Solution. Let

$$F(s) = \frac{1}{s^2+w^2}$$

Then

$$\mathscr{L}^{-1}(F(s)) = \mathscr{L}^{-1}\left(\frac{1}{s^2+w^2}\right) = \frac{\sin wt}{w}$$

Now

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{1}{s(s^2+w^2)}\right] &= \mathscr{L}^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t \mathscr{L}^{-1}[F(s)]\mathrm{d}s \\
&= \int_0^t \frac{\sin wt}{w}\mathrm{d}t = \frac{1}{w}\left(\frac{-\cos wt}{w}\right)_0^t \\
&= \frac{-1}{w}(\cos wt - 1) = \frac{1-\cos wt}{w^2}
\end{aligned}$$

Let $G(s) = \frac{1}{s(s^2+w^2)}$, so $\frac{G(s)}{s} = \frac{1}{s^2(s^2+w^2)}$.

$$\begin{aligned}
\therefore \mathscr{L}^{-1}\left[\frac{1}{s^2(s^2+w^2)}\right] &= \mathscr{L}^{-1}\left[\frac{G(s)}{s}\right] = \int_0^t \mathscr{L}^{-1}[G(s)]\mathrm{d}t \\
&= \int_0^t \frac{1-\cos wt}{w^2}\mathrm{d}t = \frac{1}{w^2}\left[t - \frac{\sin wt}{w}\right]_0^t \\
&= \frac{1}{w^2}\left[t - \frac{\sin wt}{w}\right].
\end{aligned}$$

EXERCISE

Find the inverse transform of the following

(i)	$\dfrac{1}{s^2+4s}$	(ii)	$\dfrac{1}{s^3-s}$	(iii)	$\dfrac{9}{s^2}\left(\dfrac{s+1}{s^2+9}\right)$
(iv)	$\dfrac{1}{s(s^2+1)}$	(v)	$\dfrac{s^2+3}{s(s^2+9)}$	(vi)	$\dfrac{1}{s(s+a)}$
(vii)	$\dfrac{s^2+2}{s(s^2+4)}$	(viii)	$\dfrac{1}{s^2(s+1)}$	(ix)	$\dfrac{1}{s(s^2-16)}$

Answers

(i)	$\frac{1}{4}(1-e^{-4t})$	(ii)	$\cosh t - 1$
(iii)	$1+t-\cos 3t - \frac{1}{3}\sin 3t$	(iv)	$1-\cos t$
(v)	$\frac{1}{3}[1+2\cos 3t]$	(vi)	$\dfrac{1-e^{-at}}{a}$
(vii)	$\cos^2 t$	(viii)	$t-1+e^{-t}$
(ix)	$\frac{1}{16}[\cosh 4t - 1]$		

DIFFERENTIAL EQUATIONS, INITIAL VALUE PROBLEMS

Let us now discuss how the Laplace transform method solves ODEs and initial value problems. We consider an initial value problem

$$y'' + ay' + by = r(t), \quad y(0) = K_0, y'(0) = K_1 \tag{4}$$

where a and b are constants. Here $r(t)$ is the given input (driving force) applied to the mechanical or electrical system and $y(t)$ is the output (response to the input) to be obtained. In Laplace's method we do three steps:

Step 1: Apply Laplace Transform to (4) and obtain

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where $Y = \mathscr{L}(y), R(s) = \mathscr{L}(r)$. From this we get the subsidiary equation

$$(s^2 + as + b)Y = (s+a)y(0) + y'(0) + R(s).$$

Step 2: Using completing square method, rewrite

$$s^2+as+b=\left(s+\frac{a}{2}\right)^2+\left(b-\frac{a^2}{4}\right)$$

and so we get

$$Y=\frac{(s+a)y(0)+y'(0)+R(s)}{\left(s+\frac{a}{2}\right)^2+\left(b-\frac{a^2}{4}\right)}$$

Step 3: Reduce RHS of the previous equation (usually by partial fractions as in calculus) to a sum of terms whose inverses can be found, so that we obtain the solution $y(t)=\mathscr{L}^{-1}(Y)$.

The method will be clear from the following examples.

Problem 3.3

Using Laplace transforms, find the solution of the initial value problem $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+5y=e^{-t}\sin t$, given $y(0)=0$, $y'(0)=1$.

Solution. Given equation is $y''+2y'+5y=e^{-t}\sin t$. Taking Laplace transforms of both sides,

$$\mathscr{L}(y'')+2\mathscr{L}(y')+5\mathscr{L}(y)=\mathscr{L}(e^{-t}\sin t). \tag{1}$$

But $\mathscr{L}(y')=s\mathscr{L}(y)-y(0)$ and $\mathscr{L}(y'')=s^2\mathscr{L}(y)-sy(0)-y'(0)$.
Since $y(0)=0$ and $y'(0)=1$, $\mathscr{L}(y')=s\mathscr{L}(y)$ and $\mathscr{L}(y'')=s^2\mathscr{L}(y)-1$.

$$\begin{aligned}
\text{Also } \mathscr{L}(\sin t) &= \frac{1}{s^2+1}\\
\text{So } \mathscr{L}[e^{-t}\sin t] &= \frac{1}{(s+1)^2+1} \quad \text{(by shifting theorem)}\\
&= \frac{1}{s^2+2s+2}
\end{aligned}$$

Substituting the above values in (1) we get

$$\begin{aligned}
[s^2\mathscr{L}(y)-1]+2[s\mathscr{L}(y)]+5\mathscr{L}(y) &= \frac{1}{s^2+2s+2}\\
\text{i.e. } (s^2+2s+5)\mathscr{L}(y) &= \frac{1}{s^2+2s+2}+1\\
&= \frac{1+s^2+2s+2}{s^2+2s+2}\\
&= \frac{s^2+2s+3}{s^2+2s+2}\\
\therefore \mathscr{L}(y) &= \frac{(s^2+2s+3)}{(s^2+2s+2)(s^2+2s+5)}
\end{aligned}$$

On resolving the RHS into partial fractions, we get

$$\begin{aligned}
\mathscr{L}(y) &= \frac{1/3}{s^2+2s+2} + \frac{2/3}{s^2+2s+5} \\
&= \frac{1/3}{(s+1)^2-1+2} + \frac{2/3}{(s+1)^2-1+5} \\
&= \frac{1/3}{(s+1)^2+1} + \frac{2/3}{(s+1)^2+4} \\
\therefore\ y &= \frac{1}{3}\mathscr{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right] + \frac{2}{3}\mathscr{L}^{-1}\left[\frac{1}{(s+1)^2+2^2}\right] \\
&= \frac{1}{3}e^{-t}\cdot\sin t + \frac{2}{3}\times e^{-t}\frac{\sin 2t}{2} \\
&= \frac{1}{3}[e^{-t}\sin t + e^{-t}\sin 2t].
\end{aligned}$$

■

Problem 3.4

Using Laplace transforms, solve $y'' + 3y' + 2y = 8\cos 2t$, given that $y(0) = -1$, $y'(0) = 2$.

Solution. Given equation is $y'' + 3y' + 2y = 8\cos 2t$. Taking Laplace transforms of both sides

$$\begin{aligned}
\mathscr{L}(y'') + 3\mathscr{L}(y') + 2\mathscr{L}(y) &= 8\mathscr{L}(\cos 2t) \\
[s^2\mathscr{L}(y) - sy(0) - y'(0)] + 3[s\mathscr{L}(y) - y(0)] + 2\mathscr{L}(y) &= 8\frac{s}{s^2+4} \\
(s^2\mathscr{L}(y) + s - 2 + 3sLy) + 3 + 2\mathscr{L}(y) &= \frac{8s}{s^2+4} \\
\therefore y(0) = -1, y'(0) &= 2
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}(y)[s^2+3s+2] &= \frac{8s}{s^2+4} - s - 1 \\
\text{i.e. } \mathscr{L}(y)(s^2+3s+2) &= \frac{8s - s^3 - 4s - s^2 - 4}{s^2+4} \\
\text{i.e. } \mathscr{L}(y) &= \frac{-s^3 - s^2 + 4s - 4}{(s^2+4)(s^2+3s+2)} \qquad (1)
\end{aligned}$$

Let

$$\begin{aligned}
\frac{-s^3 - s^2 + 4s - 4}{(s^2+4)(s^2+3s+2)} &= \frac{-s^3 + -s^2 4s - 4}{(s^2+4)(s+1)(s+2)} \\
&= \frac{As+B}{s^2+4} + \frac{C}{s+1} + \frac{D}{s+2}
\end{aligned}$$

Multiplying both sides by $(s^2+4)(s+1)(s+2)$, we get

$$-s^3 - s^2 + 4s - 4 = (As+B)(s+1)(s+2) + C(s^2+4)(s+2) + D(s^2+4)(s+1).$$

Put $s=-1$,

$$\begin{aligned}1-1-4-4 &= c(1+4)(-1+2)\\ -8 &= 5c\\ \Rightarrow c &= -8/5\end{aligned}$$

Put $s=-2$,

$$\begin{aligned}+8-4-8-4 &= D(4+4)(-2+1)\\ -8 &= -8D\\ \Rightarrow D &= 1\end{aligned}$$

Equating the coefficients of s^3,

$$-1=A+C+D \Rightarrow A=-1-C-D-=-1+\frac{8}{5}-1=\frac{-2}{5}$$

Equating the constant terms,

$$\begin{aligned}&-4=2B+8C+4D\\ &\Rightarrow 2B=-4-8C-4D=-4+\frac{64}{5}-4=\frac{24}{5}\\ &\Rightarrow B=\frac{12}{5}\\ &\therefore \mathscr{L}(y)=\frac{\frac{-2}{5}s+\frac{12}{5}}{s^2+4}+\frac{-8/5}{s+1}+\frac{1}{s+2}\\ &=\frac{-2}{5}\frac{s}{s^2+4}+\frac{12}{5}\frac{1}{s^2+4}-\frac{8}{5}\frac{1}{s+1}+\frac{1}{s+2}\\ &\therefore y=-\frac{2}{5}\mathscr{L}^{-1}\left(\frac{s}{s^2+4}\right)+\frac{12}{5}\mathscr{L}^{-1}\left(\frac{1}{s^2+4}\right)-\frac{8}{5}\mathscr{L}^{-1}\left(\frac{1}{s+1}\right)+\mathscr{L}^{-1}\left(\frac{1}{s+2}\right)\\ &=-\frac{2}{5}\cos 2t+\frac{12}{5}\frac{\sin 2t}{2}-\frac{8}{5}e^{-t}+e^{-2t}.\end{aligned}$$

Problem 3.5

Using Laplace transform method solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+6y=6te^{-t}$, given that $y(0)=2$, $y'(0)=5$.

Solution. Given equation is $y''+2y'+6y=6te^{-t}$. Taking Laplace transform of both sides,

$$\begin{aligned}\mathscr{L}(y'')+2\mathscr{L}(y')+6\mathscr{L}(y) &= 6\mathscr{L}(e^{-t}t)\\ [s^2\mathscr{L}(y)-sy(0)-y'(0)]+2[s\mathscr{L}(y)-y(0)]+6\mathscr{L}(y) &= 6\frac{1}{(s+1)^2}\\ s^2\mathscr{L}(y)-2s-5+2s\mathscr{L}(y)-4+6\mathscr{L}(y) &= \frac{6}{(s+1)^2}\end{aligned}$$

$$
\begin{aligned}
(s^2+2s+6)\mathscr{L}(y) =& \frac{6}{(s+1)^2}+2s+9\\
=& \frac{2s^3+13s^2+20s+15}{(s+1)^2}\\
\therefore \mathscr{L}(y) =& \frac{2s^3+13s^2+20s+15}{(s+1)^2(s^2+2s+6)} \qquad (1)
\end{aligned}
$$

Let $\frac{2s^3+13s^2+20s+15}{(s+1)^2(s^2+2s+6)} = \frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2+2s+6}$. Multiplying both sides by $(s+1)^2(s^2+2s+6)$, we get

$$2s^3+13s^2+20s+15 = A(s+1)(s^2+2s+6)+B(s^2+2s+6)+(cs+D)(s+1)^2.$$

Put $s=-1$,

$$
\begin{aligned}
-2+13-20+15 &= B(1-2+6)\\
6 &= 5B \Rightarrow B = 6/5
\end{aligned}
$$

Equating the coefficients of s^3

$$2 = A+C \qquad (2)$$

Equating the coefficients of s^2

$$13 = 3A+B+2C+D \qquad (3)$$

Equating the coefficients of s,

$$20 = 8A+2B+C+2D \qquad (4)$$

Equating the coefficients terms

$$15 = 6A+6B+D \qquad (5)$$

Solving (2), (3) and (4), (5) we get

$$
\begin{aligned}
A=0,\quad C=2,\quad D=39/5 &\\
\text{Therefore}\frac{2s^3+13s^2+20s+15}{(s+1)^2(s^2+2s+6)} &= \frac{6/5}{(s+1)^2}+\frac{2s+39/5}{s^2+2s+6}\\
&= \frac{6/5}{(s+1)^2}+\frac{2(s+1)-2+39/5}{(s+1)^2+5}\\
&= \frac{6/5}{(s+1)^2}+2\frac{(s+1)}{(s+1)^2+5}+\frac{29/5}{(s+1)^2+5}
\end{aligned}
$$

Substituting in (1),

$$\begin{aligned}
\mathscr{L}(y) &= \frac{6}{5}\cdot\frac{1}{(s+1)2}+2\frac{s+1}{(s+1)^2+5}+\frac{29}{5}\frac{1}{(s+1)^2+5}\\
\text{or } y &= \frac{6}{5}\mathscr{L}^{-1}\left(\frac{1}{(s+1)^2}\right)+2\mathscr{L}^{-1}\left(\frac{s+1}{(s+1)^2+5}\right)+\frac{29}{5}\mathscr{L}^{-1}\left(\frac{1}{(s+1)^2+5}\right)\\
&= \frac{6}{5}te^{-t}+2e^{-t}\cos\sqrt{5}t+\frac{29}{5}\times e^{-t}\frac{\sin\sqrt{5}t}{\sqrt{5}}\\
&= \frac{e^{-t}}{5}\left(6t+10\cos\sqrt{5}t+\frac{29}{\sqrt{5}}\sin\sqrt{5}t\right).
\end{aligned}$$

■

Problem 3.6

Solve the differntial equation by using Laplace transform $\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=4$, given $y(0)=2$, $y'(0)=3$.

Solution. Given equation is $y''-3y'+2y=4$. Taking Laplace transforms of both sides,

$$\begin{aligned}
\mathscr{L}(y'')-3\mathscr{L}(y')+2\mathscr{L}(y) &= \mathscr{L}(4)\\
\{s^2\mathscr{L}(y)-sy(0)-y'(0)\}-3[s\mathscr{L}(y)-y(0)]+2\mathscr{L}(y) &= 4\mathscr{L}(1)\\
s^2\mathscr{L}(y)-2s-3-3s\mathscr{L}(y)+6+2\mathscr{L}(y) &= \frac{4}{s}
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}(y)[s^2-3s+2] &= \frac{4}{s}+2s-3=\frac{2s^2-3s+4}{s}\\
\mathscr{L}(y) &= \frac{2s^2-3s+4}{s(s^2-3s+2)}=\frac{2s^2-3s+4}{s(s-1)(s-2)} \qquad (1)\\
\text{Let } \frac{2s^2-3s+4}{s(s-1)(s-2)} &= \frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-2}\\
\text{where } A &= \frac{2.0^2-3.0+4}{(0-1)(0-2)}=\frac{4}{2}=2\\
B &= \frac{2.1^2-3.1+4}{1(1-2)}=-3\\
C &= \frac{2.2^2-3\times2+4}{2(2-1)}=\frac{8-6+4}{2\times1}=3\\
\therefore \frac{2s^2-3s+4}{s(s-1)(s-2)} &= \frac{2}{s}+\frac{-3}{s-1}+\frac{3}{s-2}
\end{aligned}$$

Substituting in (1)

$$\begin{aligned}
\mathscr{L}(y) &= \frac{2}{s}-\frac{3}{s-1}+\frac{3}{s-2}\\
\therefore y &= 2\mathscr{L}^{-1}\left(\frac{1}{s}\right)-3\mathscr{L}^{-1}\left(\frac{1}{s-1}\right)+3\mathscr{L}^{-1}\left(\frac{1}{s-2}\right)\\
&= 2-3e^{t}+3e^{2t}.
\end{aligned}$$

■

EXERCISE

Solve the following differential equations by using Laplace transform.

(1) $y'' - 3y' + 2y = 4t + e^{3t}$ when $y(0) = 1,\ y'(0) = -1$

(2) $x'' - 2x' + x = e^t$, where $x(0) = 2,\ x'(0) = -1$

(3) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 3y = e^{-t},\ y(0) = y'(0) = 1$

(4) $y'' + 2y' - 3y = \sin t$, given $y(0) = y'(0) = 0$

(5) $\dfrac{d^3y}{dt^3} + 2\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} - 2y = 0$ when $y = 1,\ \dfrac{dy}{dt} = 2$, and $\dfrac{d^2y}{dt^2} = 2$ at $t = 0$.

(6) $y'' + 2y' + y = te^{-t}$ if $y(0) = 1,\ y'(0) = -2$

(7) $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$, if $y(0) = y'(0) = 0$ and $y''(0) = 6$.

(8) $y'' + 2y' + y = e^{-t}$ if $y(0) = 0$ and $y'(0) = 1$

(9) $y''' - y' = 2\cos t,\ y(0) = 3,\ y'(0) = 2$ and $y''(0) = 1$

(10) $\dfrac{d^2y}{dt^2} + 9y = \cos 2t,\ y(0) = 1$ and $y'(0) = \dfrac{12}{5}$

(11) $2y'' + 5y' + 2y = e^{-2t},\ y(0) = y'(0) = 1$

(12) $y'' + 2y' + 2y = 5\sin t,\ y(0) = y'(0) = 0$

ANSWERS

(1) $y = 3 + 2t + \dfrac{1}{2}(e^{3t} - e^t) - 2e^{2t}$

(2) $x = 2e^t - 3te^t + \dfrac{1}{2}t^2e^t$

(3) $y = \dfrac{1}{4}(7e^{-t} - 3e^{-3t} - 2te^{-t})$

(4) $y = \dfrac{1}{8}e^t - \dfrac{1}{40}e^{-3t} - \dfrac{1}{10}(2\sin t + \cos t)$

(5) $y = \dfrac{5}{3}e^t - e^{-t} + \dfrac{1}{3}e^{-2t}$

(6) $y = e^{-t}(1 - t + t^3/6)$

(7) $y = e^x - 3e^{-x} + 2e^{-2x}$

(8) $y = \frac{e^{-t}}{2}(t^2 + 2t)$

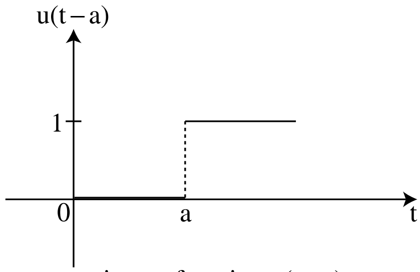
(9) $y = 2 + 2e^t - e^{-t} - \sin t$

(10) $y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$

(11) $y = \frac{20}{9}e^{-t/2} - \frac{11}{9}e^{-2t} - \frac{1}{3}te^{-2t}$

(12) $y = 2e^{-t}\cos t + e^{-t}\sin t - 2\cos t + \sin t$

4 UNIT STEP FUNCTION (HEAVISIDE FUNCTION)



unit step function u(t−a)

The unit step function $u(t-a)$ is defined as follows:

$$u(t-a) = \begin{cases} 0 & \text{when} \quad t < a \\ 1 & \text{when} \quad t \geq a \end{cases}, \qquad a \geq 0.$$

The unit step function is also called the Heaviside function.

Laplace transform of unit step function

$$\mathscr{L}[u(t-a)] = \frac{e^{-as}}{s}.$$

Proof.

By definition,
$$\begin{aligned}
\mathscr{L}[u(t-a)] &= \int_0^t e^{-st} u(t-a) \mathrm{d}t \\
&= \int_0^a e^{-st} 0 \mathrm{d}t + \int_a^\infty e^{-st} 1 \mathrm{d}t \\
&= 0 + \lim_{k\to\infty} \int_a^k e^{-st} \mathrm{d}t \\
&= \lim_{k\to\infty} \left(\frac{e^{-st}}{-s} \right)_a^k \\
&= 0 - \frac{e^{-sa}}{-s} \\
&= \frac{e^{-as}}{s}.
\end{aligned}$$

■

Problem 4.1

Express the following function in terms of unit step function and find its Laplace transform

$$f(t) = \begin{cases} 8, & t < 3 \\ 5, & t > 3 \end{cases}$$

Solution.

$$\begin{aligned}
f(t) &= \begin{cases} 8+0, & t < 3 \\ 8-3, & t > 3 \end{cases} \\
&= 8 + \begin{cases} 0, & t < 3 \\ -3, & t > 3 \end{cases} \\
&= 8 + (-3) \begin{cases} 0, & t < 3 \\ 1, & t > 3 \end{cases} \\
&= 8 - 3u(t-3) \\
\therefore \mathscr{L}[f(t)] &= \mathscr{L}(8 - 3u(t-3)] \\
&= 8\mathscr{L}(1) - 3L(u(t-3)) \\
&= \frac{8}{s} - 3 \cdot \frac{e^{-3s}}{s}
\end{aligned}$$

■

Theorem 4.1 (Second shifting theorem; t shifting)

If $f(t)$ has the Laplace transform $F(s)$, then the shifted function $\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if} \quad t < a \\ f(t-a) & \text{if} \quad t > a \end{cases}$ has the transform $e^{-as}F(s)$.

Proof.

$$\begin{aligned}
\mathscr{L}[\tilde{f}(t)] &= \mathscr{L}[f(t-a)u(t-a)] \\
&= \int_0^\infty e^{-st} f(t-a) \cdot u(t-a) \mathrm{d}t \\
&= \int_0^a e^{-st} \dot{0} \mathrm{d}t + \int_a^\infty e^{-st} f(t-a) \mathrm{d}t \\
&= \int_0^\infty e^{-st} f(t-a) \mathrm{d}t
\end{aligned}$$

Put $t-a=u$, so that $\mathrm{d}t=\mathrm{d}u$. When $t=a$, $u=0$ and $t\to\infty\Rightarrow u\to\infty$.

$$\begin{aligned}\therefore \mathscr{L}[\tilde{f}(t)] &= \int_0^\infty e^{-s(u+a)} f(u)\mathrm{d}u \\ &= e^{-as}\int_0^\infty e^{-su} f(u)\mathrm{d}u \\ &= e^{-as}F(s).\end{aligned}$$

- Since $\mathscr{L}[f(t-a)u(t-a)]=e^{-as}F(s)$, $\mathscr{L}^{-1}[e^{-as}F(s)]=f(t-a)u(t-a)$.
- Any piecewise continuous function

$$f(t)=\begin{cases} f_0(t) & \text{if} \quad 0<t<t_1 \\ f_1(t) & \text{if} \quad t_1<t<t_2 \\ \quad\vdots & \\ f_{n-1}(t) & \text{if} \quad t_{n-1}<t<t_n \\ f_n(t) & \text{if} \quad t_n<t<\infty. \end{cases}$$

defined on $0<t<\infty$ can be given by the single expression

$$\begin{aligned} f(t) = f_0(t)[u(t-0)-u(t-t_1)] + f_1(t)[u(t-t_1)-u(t-t_2)]+\cdots \\ + f_{n-1}(t)[u(t-t_{n-1})-u(t-t_n)] + f_n(t)\, u(t-t_n). \end{aligned}$$

- Effects of the unit step function

$$u(t-a)=\begin{cases} 0 & \text{if} \quad t<a, \\ 1 & \text{if} \quad t>a. \end{cases}$$

$$f(t)u(t-a)=\begin{cases} 0 & \text{if} \quad t<a, \\ f(t) & \text{if} \quad t>a. \end{cases}$$

and

$$f(t-a)u(t-a)=\begin{cases} 0 & \text{if} \quad t<a, \\ f(t-a) & \text{if} \quad t>a. \end{cases}$$

Problem 4.2

Express the following function in terms of unit step function and hence find its Laplace transform

$$f(t)=\begin{cases} 2 & \text{if} \quad 0<t<\pi \\ 0 & \text{if} \quad \pi<t<2\pi \\ \sin t & \text{if} \quad t>2\pi. \end{cases}$$

Solution. The given function $f(t)$ can be expressed as

$$\begin{aligned} f(t) &= 2[u(t-0)-u(t-\pi)]+0[u(t-\pi)-u(t-2\pi)]+\sin t\cdot u(t-2\pi) \\ &= 2u(t)-2u(t-\pi)+\sin t\cdot u(t-2\pi) \\ &= 2u(t)-2u(t-\pi)+\sin(t-2\pi)\cdot u(t-2\pi) \\ \therefore \mathscr{L}(f(t)) &= \frac{2}{s}-\frac{2e^{-\pi s}}{s}+\frac{e^{-2\pi s}}{s^2+1}. \end{aligned}$$

Problem 4.3

Express the following function in terms of unit step function

$$f(t)=\begin{cases} 2+t^2 & \text{if} \quad 0<t<2 \\ 6 & \text{if} \quad 2<t<3 \\ \dfrac{2}{2t-5} & \text{if} \quad t>3. \end{cases}$$

Solution. The given function $f(t)$ can be expressed as

$$\begin{aligned} f(t) &= (2+t^2)[u(t-0)-u(t-2)]+6[u(t-2)-u(t-3)]+\frac{2}{2t-5}\cdot u(t-3) \\ &= (2+t^2)u(t)+(4-t^2)\,u(t-2)+\left(\frac{32-12t}{2t-5}\right)\cdot u(t-3). \end{aligned}$$

Problem 4.4

Find the inverse transform of $F(s)=\dfrac{2}{s^2}-\dfrac{2e^{-2s}}{s^2}-\dfrac{4e^{-2s}}{s}+\dfrac{se^{-\pi s}}{s^2+1}$.

Solution. We have $\mathscr{L}^{-1}\left(\dfrac{1}{s^2}\right)=t$, $\mathscr{L}^{-1}\left(\dfrac{1}{s}\right)=1$ and $\mathscr{L}^{-1}\left(\dfrac{s}{s^2+1}\right)=\cos t$. Now

$$\mathscr{L}^{-1}[e^{-as}F(s)]=f(t-a)\cdot u(t-a)$$

where $f(t)=\mathscr{L}^{-1}(F(s))$. This shows that,

$$\begin{aligned} \mathscr{L}^{-1}[e^{-as}F(s)] &= f(t-a)\cdot u(t-2), \\ \mathscr{L}^{-1}\left[e^{-2s}\frac{1}{s}\right] &= 1\cdot u(t-2), \\ \mathscr{L}^{-1}\left[e^{-\pi s}\frac{s}{s^2+1}\right] &= \cos(t-\pi)u(t-\pi) \\ \mathscr{L}^{-1}\left[e^{-2s}\frac{1}{s^2}\right] &= (t-2)u(t-2). \end{aligned}$$

Thus

$$\begin{aligned}
\mathscr{L}^{-1}[F(s)] &= \mathscr{L}^{-1}\left[\frac{2}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{4e^{-2s}}{s} + \frac{se^{-\pi s}}{s^2+1}\right] \\
&= 2\mathscr{L}^{-1}\left(\frac{1}{s^2}\right) - 2\mathscr{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right) - 4\mathscr{L}^{-1}\left[\frac{e^{-2s}}{s}\right) + \mathscr{L}^{-1}\left[\frac{se^{-\pi s}}{s^2+1}\right] \\
&= 2t - 2(t-2)u(t-2) - 4u(t-2) + \cos(t-\pi)u(t-\pi) \\
&= 2t - 2tu(t-2) - \cos t u(t-\pi) \\
&= \begin{cases} 2t & \text{if} \quad 0 < t < 2 \\ 0, & \text{if} \quad 2 < t < \pi \\ -\cos t & \text{if} \quad t > \pi \end{cases}
\end{aligned}$$

■

Problem 4.5

Find the inverse Laplace transform of the following:

(i) $\dfrac{e^{-\pi s}}{s^2+1}$

(ii) $\dfrac{se^{-s/2} + \pi e^{-s}}{s^2+\pi^2}$

(iii) $\dfrac{3}{s} - \dfrac{4e^{-s}}{s^2} + \dfrac{4e^{-3s}}{s^2}$

Solution. (i) We have $\mathscr{L}^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t$. Since $\mathscr{L}^{-1}[e^{-as}F(s)] = f(t-a)u(t-a)$ where $f(t) = \mathscr{L}^{-1}[F(s)]$,

$$\begin{aligned}
\mathscr{L}^{-1}\left[e^{-\pi s}\frac{1}{s^2+1}\right] &= \sin(t-\pi)\cdot u(t-\pi) \\
&= -\sin t \cdot u(t-\pi)
\end{aligned}$$

(ii) $$\frac{se^{-\frac{1}{2}s} + \pi e^{-s}}{s^2+\pi^2} = e^{-\frac{1}{2}s}\frac{s}{s^2+\pi^2} + e^{-s}\frac{\pi}{s^2+\pi^2}. \tag{1}$$

Now $\mathscr{L}^{-1}\left[\dfrac{s}{s^2+\pi^2}\right] = \cos\pi t$ and $\mathscr{L}^{-1}\left[\dfrac{\pi}{s^2+\pi^2}\right] = \sin\pi t$.

By second shifting property,

$$\begin{aligned}
\mathscr{L}^{-1}\left[e^{-\frac{1}{2}s}\frac{s}{s^2+\pi^2}\right] &= \cos\pi\left(t-\frac{1}{2}\right)\cdot u\left(t-\frac{1}{2}\right) \\
&= \sin\pi t\cdot u\left(t-\frac{1}{2}\right) \\
\text{and } \mathscr{L}^{-1}\left[e^{-s}\cdot\frac{\pi}{s^2+\pi^2}\right] &= \sin\pi(t-1)u(t-1) \\
&= -\sin\pi t\cdot u(t-1).
\end{aligned}$$

$$\text{By (1),}\quad \mathscr{L}^{-1}\left[\frac{se^{-\frac{1}{2}s}+\pi e^{-s}}{s^2+\pi^2}\right]$$
$$= \sin\pi t\cdot u\left(t-\frac{1}{2}\right)-\sin\pi t\cdot u(t-1)$$
$$= \sin\pi t\left[u\left(t-\frac{1}{2}\right)-u(t-1)\right]$$

(iii) Let $F(s)=\frac{3}{s}-4e^{-s}\cdot\frac{1}{s^2}+4e^{-3s}\frac{1}{s^2}$. (1)

Now $\mathscr{L}^{-1}\left[\frac{1}{s}\right]=1$, $\mathscr{L}^{-1}\left(\frac{1}{s^2}\right)=t$.

By second shifting properties,

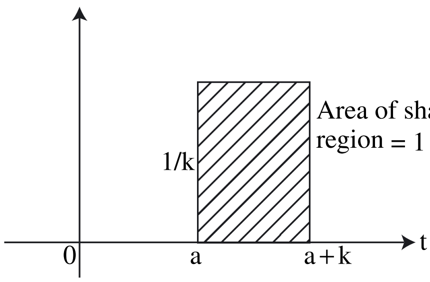
$$\mathscr{L}^{-1}\left[e^{-s}\frac{1}{s^2}\right] = (t-1)u(t-1) \quad \text{and}$$
$$\mathscr{L}^{-1}\left[e^{-3s}\frac{1}{s^2}\right] = (t-3)u(t-3).$$

By (1), $\mathscr{L}^{-1}[F(s)]=3-4(t-1)u(t-1)+4(t-3)u(t-3)$. ■

5 UNIT IMPULSE FUNCTION

The unit impulse function is considered as the limiting form of the function of

$$\delta_k(t-a)=\begin{cases}1/k & \text{if}\quad a\le t\le a+k\\ 0 & \text{otherwise.}\end{cases} \tag{5}$$



This function is represented in the figure. In mechanics, the impuse of a force $f(t)$ over a time interval, $a\le t\le a+k$ is defined to be the integral of $f(t)$ from a to $a+k$. Thus the impulse I_k of the function (1) is

$$I_k=\int_a^{a+k}\delta_k(t-a)\mathrm{d}t$$
$$=\int_a^{a+k}\frac{1}{k}\mathrm{d}t$$
$$=\frac{1}{k}(t)_a^{a+k}$$
$$=\frac{1}{k}\times k=1.$$

The limit of $\delta_k(t-a)$ as $k \to 0$ $(k>0)$ is denoted by $\delta(t-a)$ and it is called the unit impuse function or Dirac delta function. Thus the unit impulse function $\delta(t-a)$ is defined as follows.

$$\delta(t-a) = \begin{cases} \infty & \text{if} \quad t=a \\ 0 & \text{if} \quad t \neq a. \end{cases} \qquad \text{and} \qquad \int_0^\infty \delta(t-a)\mathrm{d}t = 1.$$

By the definition, the Laplace transform of $\delta(t-a)$ can be obtained as

$$\begin{aligned}
\mathscr{L}[\delta(t-a)] &= \int_0^\infty e^{-st}\delta(t-a)dt \\
&= \lim_{k\to 0}\int_0^\infty e^{-st}\delta_k(t-a)dt \\
&= \lim_{k\to 0}\int_a^{a+k} e^{-st}\frac{1}{k}dt \\
&= \lim_{k\to 0}\frac{1}{k}\left\{\frac{e^{-st}}{-s}\right\}_a^{a+k} \\
&= \lim_{k\to 0}\frac{e^{-s(a+k)}-e^{-as}}{-ks} \\
&= \lim_{k\to 0}\frac{e^{-as}(1-e^{-ks})}{ks} = \frac{e^{-as}}{s}\lim_{k\to 0}\frac{1-e^{-sk}}{k} \\
&= \frac{e^{-as}}{s}\lim_{k\to 0}\frac{e^{-sk}s}{1} \quad \text{(by L'Hospitals rule)} \\
&= e^{-as}.
\end{aligned}$$

When $a=0$, $\mathscr{L}[\delta(t)] = e^0 = 1$.

6 PERIODIC FUNCTIONS

If $f(t)$ is a periodic function with period T, then

$$\mathscr{L}[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T \mathrm{e}^{-st}f(t)\mathrm{d}t \quad \text{for} s>0. \tag{6}$$

Proof. By definition,

$$\begin{aligned}
\mathscr{L}[f(t)] &= \int_0^\infty e^{-st}f(t)\mathrm{d}t \\
&= \int_0^T e^{-st}f(t)\mathrm{d}t + \int_T^{2T} e^{-st}f(t)\mathrm{d}t + \int_{2T}^{3T} e^{-st}f(t)\mathrm{d}t + \cdots.
\end{aligned}$$

Substituting $t=u+T$ in second integral and $t=u+2T$ in third integral and so on.

$$\begin{aligned}
\mathscr{L}[f(t)] &= \int_0^T e^{-st}f(t)\mathrm{d}t + \int_0^T e^{-s(u+T)}f(u+T)\mathrm{d}u + \int_0^T e^{-s(u+2T)}f(u+2T)\mathrm{d}u + \cdots \\
&= \int_0^T e^{-su}f(u)\mathrm{d}u + e^{-sT}\int_0^T e^{-su}f(u)\mathrm{d}u + e^{-2sT}\int_0^T e^{-su}f(u)\mathrm{d}u + \cdots \\
&= (1+e^{-sT}+e^{-2sT}+\cdots)\int_0^T e^{-su}f(u)\mathrm{d}u \\
&= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\mathrm{d}t \quad \left(\because 1+x+x^2+\cdots = \frac{1}{1-x}\right).
\end{aligned}$$

■

Problem 6.1

Find the Laplace transform of the saw-toothed wave function of period T, defined by $f(t) = \frac{kt}{T}, 0 < t < T$.

Solution. We have

$$\begin{aligned}
\mathscr{L}[f(t)) &= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) \mathrm{d}t \\
&= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \frac{kt}{T} \mathrm{d}t \\
&= \frac{k}{T(1-e^{-sT})} \int_0^T t e^{-st} \mathrm{d}t \\
&= \frac{k}{T(1-e^{-sT})} \left[\left(t \frac{e^{-st}}{-s} \right)_0^T - \int_0^T 1 \frac{e^{-st}}{-s} \mathrm{d}t \right] \\
&= \frac{k}{T(1-e^{-sT})} \left[\frac{Te^{-sT}}{-s} + \frac{1}{s} \left(\frac{e^{-st}}{-s} \right)_0^T \right] \\
&= \frac{k}{T(1-e^{-sT})} \left[\frac{Te^{-sT}}{-s} - \frac{1}{s^2} (e^{-sT} - 1) \right].
\end{aligned}$$

■

Problem 6.2

Find the Laplace transform of the triangular wave function of period $2a$ given by

$$f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases}$$

Solution.

$$\begin{aligned}
\mathscr{L}(f(t)] &= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) \mathrm{d}t \\
&= \frac{1}{1-e^{-s \cdot 2a}} \int_0^{2a} e^{-st} f(t) \mathrm{d}t \\
&= \frac{1}{1-e^{-2as}} \left\{ \int_0^a e^{-st} t \mathrm{d}t + \int_a^{2a} e^{-st} (2a - t) \mathrm{d}t \right\} \\
&= \frac{1}{1-e^{-2as}} \left\{ \left[t \cdot \frac{e^{-st}}{-s} - 1 \left(\frac{e^{-st}}{s^2} \right) \right]_0^a - \left[(2a-t) \frac{e^{-st}}{-s} - (-1) \left(\frac{e^{-st}}{s^2} \right) \right]_a^{2a} \right\} \\
&= \frac{1}{s^2} \tanh \frac{as}{2}.
\end{aligned}$$

■

Problem 6.3

Find the Laplace transform of the Half-wave rectifier function

$$f(t) = \begin{cases} \sin wt & for \quad 0 < t < \pi/w \\ 0 & for \quad \pi/w < t < \dfrac{2\pi}{w} \end{cases}$$

Solution. $f(t)$ is a periodic function with period $T = \dfrac{2\pi}{w}$.

$$\begin{aligned} \text{Let} \quad \mathscr{L}[f(t)] &= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt \\ &= \frac{1}{1-e^{-s(\frac{2\pi}{w})}} \left[\int_0^{\pi/w} e^{-st} \sin wt dt + \int_{\pi/w}^{2\pi/w} e^{-st} o dt \right] \\ &= \frac{1}{1-e^{-s(\frac{2\pi}{w})}} \left\{ \frac{e^{-st}}{s^2+w^2} (-s \sin wt - w \cos wt) \right]_0^{\pi/w} \\ &= \frac{1}{1-e^{\frac{-2\pi s}{w}}} \left\{ \frac{e^{-s\pi/w} \cdot \omega}{s^2+w^2} + \frac{w}{s^2+w^2} \right\}. \end{aligned}$$

■

7 CONVOLUTION

The convolution of f and g is denoted by $f*g$ and is defined as

$$f*g = \int_0^t f(u) g(t-u) du. \tag{7}$$

Also $f*g = g*f$.

Theorem 7.1 (Convolution Theorem)

If two functions f and g satisfy the assumption in the existence theorem, so that their transforms F and G exist, the product $H = FG$ is the transform of $h = f*g$. From this

$$\mathscr{L}^{-1}(F(s)G(s)) = f*g = \int_0^t f(u) g(t-u) du.$$

Problem 7.1

Apply convolution theorem to evaluate the inverse Laplace transform of the following:

(i) $\dfrac{s}{(s^2+a^2)^2}$

(ii) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

(iii) $\dfrac{1}{(s^2+a^2)^2}$

(iv) $\dfrac{s^2}{(s^2+4)^2}$

(v) $\dfrac{1}{s(s^2-a^2)}$

Solution. (1) Let $\dfrac{s}{(s^2+a^2)^2} = \dfrac{s}{s^2+a^2} \times \dfrac{1}{s^2+a^2} = f_1(s)f_2(s)$ where $f_1(s) = \dfrac{s}{s^2+a^2}$ and $f_2(s) = \dfrac{1}{s^2+a^2}$.

Now

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}[f_1(s)] \\ &= \mathscr{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at \end{aligned}$$

and

$$\begin{aligned} g(t) &= \mathscr{L}^{-1}[f_2(s)] \\ &= \mathscr{L}^{-1}[\frac{1}{s^2+a^2}] \\ &= \frac{\sin at}{a}. \end{aligned}$$

By convolution theorem,

$$\begin{aligned} \mathscr{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] &= \mathscr{L}^{-1}[f_1(s)f_2(s)] \\ &= \int_0^t f(t-u)g(u)\mathrm{d}u \\ &= \int_0^t \cos a(t-u)\frac{\sin au}{a}\mathrm{d}u \\ &= \frac{1}{2a}\int_0^t [\sin at + \sin(2au-at)]\mathrm{d}u \\ &= \frac{1}{2a}\left[u\sin at - \frac{1}{2a}\cos(2au-at)\right]_0^t \\ &= \frac{1}{2a}t\sin at. \end{aligned}$$

(2) Let $f_1(s) = \frac{s}{s^2+a^2}$ and $f_2(s) = \frac{s}{s^2+b^2}$

$$f(t) = \mathscr{L}^{-1}(f_1(s)) = \mathscr{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$
$$g(t) = \mathscr{L}^{-1}(f_2(s)) = \mathscr{L}^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt$$

$$\begin{aligned}
\therefore \mathscr{L}^{-1}&\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] \\
&= \mathscr{L}^{-1}[f_1(s)f_2(s)] \\
&= \int_0^t f(t-u)g(u)\mathrm{d}u, \text{ by convolution theorem} \\
&= \int_0^t \cos a(t-u)\cos bu\mathrm{d}u \\
&= \frac{1}{2}\int_0^t [\cos(at-(a-b)u)+\cos(at-(a+b)u)]\mathrm{d}u \\
&= \frac{1}{2}\left[\frac{\sin(at-(a-b)u)}{-(a-b)}+\frac{\sin(at-(a+b)u)}{-(a+b)}\right]_0^t \\
&= -\frac{1}{2}\left[\frac{\sin bt-\sin at}{a-b}-\frac{(\sin at+\sin bt)}{a+b}\right] \\
&= \frac{a\sin at-b\sin bt}{a^2-b^2}
\end{aligned}$$

(3) Let $f_1(s) = \frac{1}{s^2+a^2}$ and $f_2(s) = \frac{1}{s^2+a^2}$.

Then

$$\begin{aligned}
f(t) &= \mathscr{L}^{-1}[f_1(s)] \\
&= \mathscr{L}^{-1}\left[\frac{1}{s^2+a^2}\right] \\
&= \frac{\sin at}{a} \\
g(t) &= \mathscr{L}^{-1}[f_2(s)] \\
&= \mathscr{L}^{-1}\left[\frac{1}{(s^2+a^2)}\right] \\
&= \frac{\sin at}{a}.
\end{aligned}$$

By convolution theorem, we have

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{1}{(s^2+a^2)}\right] &= \mathscr{L}^{-1}[f_1(s)f_2(s)] \\
&= \int_0^t f(t-u)g(u)\mathrm{d}u \\
&= \int_0^t \frac{\sin a(t-u)}{a}\cdot\frac{\sin au}{a}\mathrm{d}u \\
&= \frac{1}{a^2}\int_0^t \sin(at-au)\cdot\sin au\mathrm{d}u \\
&= \frac{1}{2a^2}\int_0^t [\cos(at-2au)-\cos(at)]\mathrm{d}u \\
&= \frac{1}{2a^2}\left\{\frac{\sin(at-2au)}{-2a}-u\cos at\right\}_{u=0}^{t} \\
&= \frac{1}{2a^2}\times\left[\frac{2\sin at}{2a}-t\cos at\right] \\
&= \frac{1}{2a^3}[\sin at - at\cos at]
\end{aligned}$$

(4) Let $f_1(s)=\frac{s}{s^2+4}$ and $f_2(s)=\frac{s}{s^2+4}$.

Then $f(t)=\mathscr{L}^{-1}[f_1(s)]=\mathscr{L}^{-1}\left[\frac{s}{s^2+2^2}\right]=\cos 2t$

and $g(t)=\mathscr{L}^{-1}[f_2(s)]=\cos 2t$

By convolution theorem, we have

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{s^2}{(s^2+4)^2}\right] &= \mathscr{L}^{-1}[f_1(s)\cdot f_2(s)] \\
&= \int_0^t f(t-u)\cdot g(u)\mathrm{d}u \\
&= \int_0^t \cos(2t-2u)\cdot\cos 2u\mathrm{d}u \\
&= \frac{1}{2}\int_0^t [\cos 2t+\cos(2t-4u)]\mathrm{d}u \\
&= \frac{1}{2}\left\{u\cos 2t+\frac{\sin(2t-4u)}{-4}\right\}_{u=0}^{t} \\
&= \frac{1}{2}\left\{t\cos 2t+\frac{\sin 2t}{2}\right\}
\end{aligned}$$

(5) Let $f_1(s)=\frac{1}{s}$ and $f_2(s)=\frac{1}{s^2-a^2}$.

Then $f(t)=\mathscr{L}^{-1}[f_1(s)]=\mathscr{L}^{-1}\left(\frac{1}{s}\right)=1$ and

$g(t)=\mathscr{L}^{-1}[f_2(s)]=\mathscr{L}^{-1}\left[\frac{1}{s^2-a^2}\right]=\frac{\sinh at}{a}$.

By convolution theorem, we have

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{1}{s(s^2-a^2)}\right] &= \mathscr{L}^{-1}[f_1(s), f_2(s)] \\ &= \int_0^t f(t-u)\cdot g(u)\mathrm{d}u \\ &= \int_0^t \cdot 1 \cdot \frac{\sinh au}{a}\mathrm{d}u = \frac{1}{a}\left(\frac{\cosh au}{a}\right)_0^t \\ &= \frac{1}{a^2}(\cosh at - 1)\end{aligned}$$

■

EXERCISE

(i) Apply convolution theorem to find the inverse Laplace transform of

(i)	$\frac{1}{(s+a)(s+b)}$	(ii)	$\frac{1}{s(s^2+4)}$	(iii)	$\frac{3}{(s^2+1)(s^2+9)}$
(iv)	$\frac{s}{(s^2+1)(s^2+4)}$	(v)	$\frac{s^2}{s^4-a^4}$	(vi)	$\frac{2}{(s+1)(s^2+4)}$
(vii)	$\frac{1}{(s+1)(s+9)^2}$	(viii)	$\frac{1}{(s-2)(s-3)}$	(ix)	$\frac{1}{(s+1)(s^2+1)}$
(x)	$\frac{1}{s^2(s^2+a^2)}$				

(ii) Find the Laplace tranform of the following

(i) $t^2u(t-2)$ (ii) $\sin t \cdot u(t-4)$

(iii) Find the inverse Laplace transforms of

(i)	$\frac{se^{-as}}{s^2-w^2}$	(ii)	$\frac{e^{-2s}}{s-3}$		
(iii)	$\frac{e^{-s}}{(s-1)(s-2)}$	(iv)	$\frac{se^{-2s}}{s^2-1}$	(v)	$\frac{e^{-s}}{(s+1)^3}$

(iv) Find the Lapalce transform of the square-wave function of period a defined as $f(t) = \begin{cases} 1 & \text{for} \quad 0<t<a/2 \\ -1 & \text{for} \quad a/2<t<a \end{cases}$.

(v) Find the Laplace transform of the wave form $f(t) = \frac{2t}{3}, 0 \le t \le 3$.

(vi) Find the Laplace transform of the periodic function $f(t) = e^t$ for $0 < t < 2\pi$.

(vii) Find the Laplace transform of

$$f(t) = \begin{cases} \cos wt, & \text{for} \quad 0 < t < \pi/w \\ 0 & \text{for} \quad \pi/w < t < 2\pi/w \end{cases}$$

ANSWERS

1. (i) $\dfrac{e^{-bt} - e^{-at}}{a-b}$ (ii) $\dfrac{1}{4}(1-\cos 2t)$

   (iii) $\dfrac{1}{8}[3\sin t - \sin 3t]$ (iv) $\dfrac{1}{3}(\cos t - \cos 2t)$

   (v) $\dfrac{1}{2a}(\sinh at + \sin at)$ (vi) $\dfrac{2e^{-t}}{5} - \dfrac{1}{5}(2\cos 2t - \sin 2t)$

   (vii) $\dfrac{e^{-t}}{64}[1 - e^{-8t}(1+8t)]$ (viii) $e^{2t} - e^{3t}$

   (ix) $\dfrac{1}{2}[e^{-t} + \sin t - \cos t]$ (x) $\dfrac{1}{a^3}(at - \sin at)$

2. (i) $\dfrac{e^{-2s}}{s^3}(4s^2 + 4s + 2)$ (ii) $\dfrac{e^{-4s}}{s^2+1}(\cos 4 + s\sin 4)$

3. (i) $\cosh w(t-a) \cdot u(t-a)$ (ii) $e^{3(t-2)}u(t-2)$

   (iii) $[e^{2(t-1)} - e^{t-1}]u(t-1)$ (iv) $\cosh(t-2) \cdot u(t-2)$

   (v) $\dfrac{1}{2}e^{-(t-1)}(t-1)^2 u(t-1)$

4. $\dfrac{1}{s}\tanh\left(\dfrac{as}{4}\right)$

5. $\dfrac{2e^{-3s}}{-s(1-e^{-3s})} + \dfrac{2}{3s^2}$

6. $\dfrac{e^{2(1-s)\pi} - 1}{(1-s)(1-e^{-2s\pi})}$

7. $\dfrac{s}{(s^2+w^2)(1-e^{-\pi s/w})}$

8 DIFFERENTIATION AND INTEGRATION OF TRANSFORMS

Theorem 8.1 (Differentiation of Transforms/Multiplication by t^n)

If $\mathscr{L}[f(t)] = F(s)$, *then* $\mathscr{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}[F(s)]$ *where* $n = 1, 2, 3, \ldots$.

Proof. Given

$$F(s) = \mathscr{L}[f(t) = \int_0^\infty \mathrm{e}^{-st} \cdot f(t)\,\mathrm{d}t \tag{1}$$

Differentiating (1) w.r.t 's', we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}[F(s)] &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\int_0^\infty \mathrm{e}^{-st} \cdot f(t)\,\mathrm{d}t\right] \\
&= \int_0^\infty \frac{\partial}{\partial s}[\mathrm{e}^{-st}] f(t)\,\mathrm{d}t \\
&= \int_0^\infty -t\,\mathrm{e}^{-st} \cdot f(t)\,\mathrm{d}t \\
&= -\int_0^\infty \mathrm{e}^{-st}[t f(t)]\,\mathrm{d}t \\
&= -\mathscr{L}[t f(t)].
\end{aligned}$$

$$\text{or} \qquad \mathscr{L}[t f(t)] = (-1)^1 \frac{\mathrm{d}}{\mathrm{d}s}[F(s)] \tag{2}$$

$$\begin{aligned}
\text{Similarly,} \qquad \mathscr{L}[t^2 f(t)] &= (-1)^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2}[F(s)] \\
\mathscr{L}[t^3 f(t)] &= (-1)^3 \frac{\mathrm{d}^3}{\mathrm{d}s^3}[F(s)]. \\
&\vdots \\
\text{In general,} \qquad \mathscr{L}[t^n f(t)] &= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}[F(s)].
\end{aligned}$$

■

Problem 8.1

Find the Laplace transform of the following

(i) $t\mathrm{e}^{-t}\cos t$

(ii) $t^2 \sin wt$

(iii) $t \sinh at$

(iv) $t^2 \mathrm{e}^t \cdot \sin 4t$

(v) $t\mathrm{e}^{2t}\cos 5t$

(vi) $t \sin^2 3t$

Solution. (i) We have

$$\mathscr{L}[\cos t] = \frac{s}{s^2+1}$$

$$\therefore \quad \mathscr{L}[t\cos t] = (-1)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+1}\right)$$

$$= (-1)\frac{[(s^2+1)\cdot 1 - s\cdot 2s]}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

By shifting property, we get

$$\mathscr{L}[\mathrm{e}^{-t}t\cos t] = \{\mathscr{L}[t\cos t]\}_{s\to s+1}$$

$$= \frac{(s+1)^2-1}{[(s+1)^2+1]^2} = \frac{s^2+2s}{(s^2+2s+2)^2}$$

(ii) We have $\mathscr{L}[\sin wt] = \dfrac{w}{s^2+w^2}$

$$\therefore \quad \mathscr{L}[t^2\sin wt] = (-1)^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left[\frac{w}{s^2+w^2}\right]$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{w}{s^2+w^2}\right]\right\}$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{-2ws}{(s^2+w^2)^2}\right\}$$

$$= -2w\left\{\frac{(s^2+w^2)^2\times 1 - s\times 2(s^2+w^2)\times 2s}{(s^2+w^2)^4}\right\}$$

$$= -2w\left\{\frac{s^2+w^2-4s^2}{(s^2+w^2)^3}\right.$$

$$= \frac{6ws^2-2w^3}{(s^2+w^2)^3}$$

(iii) we have $\mathscr{L}[\sinh at] = \dfrac{a}{s^2-a^2}$

$$\therefore \quad \mathscr{L}[t\sinh at] = (-1)\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{a}{s^2-a^2}\right] = \frac{2as}{(s^2-a^2)^2}$$

(iv) we have $\mathscr{L}[\sin 4t] = \dfrac{4}{s^2+16}$

$$\therefore \quad \mathscr{L}[t^2\sin 4t] = (-1)^2\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left[\frac{4}{s^2+16}\right]$$

$$= 4\times\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+16}\right)\right]$$

$$= 4\times\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{-2s}{(s^2+16)^2}\right]$$

$$= -8\left[\frac{(s^2+16)^2 1 - s\times 2(s^2+16)\times 2s}{(s^2+16)^4}\right.$$

$$= \frac{24s^2-128}{(s^2+16)^3}$$

By shifting property, we get

$$\begin{aligned}\mathscr{L}[e^t \cdot t^2 \sin 4t] &= \{\mathscr{L}[t^2 \sin 4t]\}_{s\to s-1} \\ &= \frac{24(s-1)^2-128}{((s-1)^2+16)^3} \\ &= \frac{24s^2-48s-104}{(s^2-2s+17)^3}\end{aligned}$$

(v) we have $\mathscr{L}[\cos 5t] = \dfrac{s}{s^2+25}$

so
$$\begin{aligned}\mathscr{L}[t\cos 5t] &= (-1)\frac{d}{ds}\left[\frac{s}{s^2+25}\right] \\ &= (-1)\left[\frac{(s^2+25)-s\times 2s}{(s^2+25)^2}\right] = \frac{s^2-25}{(s^2+25)^2}\end{aligned}$$

By shifting property, we get

$$\begin{aligned}\mathscr{L}[e^{2t} t\cos 5t] &= \{\mathscr{L}[t\cos 5t]\}_{s\to s-2} \\ &= \frac{(s-2)^2-25}{((s-2)^2+25)^2} \\ &= \frac{s^2-4s-21}{(s^2-4s+29)^2}\end{aligned}$$

(vi) we have $\sin^2 3t = \dfrac{1-\cos 6t}{2}$

So

$$\begin{aligned}\mathscr{L}[\sin^2 3t] &= \frac{1}{2}[\mathscr{L}(1)-\mathscr{L}(\cos 6t)] = \frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+36}\right] \\ \therefore \quad \mathscr{L}[t\sin^2 3t] &= (-1)\frac{d}{ds}\left\{\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+36}\right)\right\} \\ &= -\frac{1}{2}\left[-\frac{1}{s^2}-\left[\frac{(s^2+36)1-s\times 2s}{(s^2+36)^2}\right]\right\} \\ &= \frac{1}{2s^2}+\frac{1}{2}\times\left[\frac{36-s^2}{(s^2+36)^2}\right]\end{aligned}$$

■

Theorem 8.2 (Integration of Transforms /Division by t)

If $\mathscr{L}[f(t)] = F(s)$, then $L\left[\dfrac{f(t)}{t}\right] = \displaystyle\int_s^\infty F(s)\,ds$.

Proof. Given $F(s) = \mathscr{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$. Integrating both sides w.r.t. s between s and ∞.

$$
\begin{aligned}
\int_s^\infty F(s)\, ds &= \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\, dt\right] ds. \\
&= \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\, ds\right] dt \\
&= \int_0^\infty \left\{ f(t) \frac{e^{-st}}{-t} \right\}_s^\infty dt \\
&= \int_0^\infty \left\{ 0 - f(t) \frac{e^{-st}}{-t} \right\} dt \\
&= \int_0^\infty e^{-st} \left[\frac{f(t)}{t}\right] dt \\
&= L\left[\frac{f(t)}{t}\right].
\end{aligned}
$$

Hence the result ■

EXAMPLE 8

Find the Laplace transform of the following.

(i) $\dfrac{1-e^t}{t}$ (ii) $\dfrac{e^{-at}-e^{-bt}}{t}$

(iii) $\dfrac{\sin t}{t}$ (iv) $\dfrac{\cos at - \cos bt}{t}$

(v) $\dfrac{1-\cos t}{t^2}$

Solution

(i) We have $\mathscr{L}[1-e^t] = \mathscr{L}(1) - \mathscr{L}(e^t) = \dfrac{1}{s} - \dfrac{1}{s-1}$

$$
\begin{aligned}
\therefore \quad L\left[\frac{1-e^t}{t}\right] &= \int_s^\infty \mathscr{L}(1-e^t)\, ds \\
&= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds = \{\log s - \log(s-1)\}_s^\infty \\
&= \left\{\log\left(\frac{s}{s-1}\right)\right\}_s^\infty = \left\{\log\left(\frac{1}{1-\frac{1}{s}}\right)\right\}_s^\infty \\
&= \log 1 - \log\left(\frac{1}{1-\frac{1}{s}}\right) \\
&= 0 - \log\left(\frac{s}{s-1}\right) = \log\left(\frac{s-1}{s}\right)
\end{aligned}
$$

(ii) we have $\mathscr{L}[e^{-at} - e^{-bt}] = \dfrac{1}{s+a} - \dfrac{1}{s+b}$

$$\begin{aligned}
\therefore \quad L\left[\frac{e^{-at} - e^{-bt}}{t}\right] &= \int_s^\infty \mathscr{L}[e^{-at} - e^{-bt}]\,ds \\
&= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds \\
&= \{\log(s+a) - \log(s+b)\}_s^\infty \\
&= \log\left(\frac{s+a}{s+b}\right)\Big|_s^\infty \\
&= \left\{\log\frac{1+a/s}{1+b/s}\right\}_s^\infty \\
&= \log 1 - \log\left(\frac{1+a/s}{1+b/s}\right) \\
&= 0 - \log\left(\frac{s+a}{s+b}\right) \\
&= \log\left(\frac{s+b}{s+a}\right)
\end{aligned}$$

(iii) we have $\mathscr{L}[\sin t] = \dfrac{1}{s^2+1}$

$$\begin{aligned}
\therefore \quad L\left[\frac{\sin t}{t}\right] &= \int_s^\infty \mathscr{L}(\sin t)\,ds \\
&= \int_s^\infty \frac{1}{s^2+1}\,ds \\
&= \tan^{-1}(s)|_s^\infty = \frac{\pi}{2} - \tan^{-1} s \\
&= \frac{\pi}{2} - \tan^{-1} s \\
&= \cot^{-1}(s) = \tan^{-1}(1/s)
\end{aligned}$$

(iv) we have $\mathscr{L}[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$$\begin{aligned}
\therefore \quad L\left[\frac{\cos at - \cos bt}{t}\right] &= \int_s^\infty \mathscr{L}[\cos at - \cos bt]\,\mathrm{d}s \\
&= \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]\mathrm{d}s \\
&= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty \\
&= \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\Bigg|_s^\infty \\
&= \frac{1}{2}\log\frac{1+a^2/s^2}{1+b^2/s^2}\Bigg|_s^\infty \\
&= \frac{1}{2}\left[\log 1 - \log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right)\right] \\
&= \frac{1}{2}\left[0 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] \\
&= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}
\end{aligned}$$

(v) we have $\mathscr{L}[1-\cos t] = \frac{1}{s} - \frac{s}{s^2+1}$

$$\begin{aligned}
\therefore \quad L\left[\frac{1-\cos t}{t}\right] &= \int_s^\infty L[1-\cos t]\,\mathrm{d}s \\
&= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right]\mathrm{d}s \\
&= \left\{\log s - \frac{1}{2}\log(s^2+1)\right\}_s^\infty \\
&= \frac{1}{2}[\log s^2 - \log(s^2+1)]_s^\infty \\
&= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+1}\right)\right]_s^\infty \\
&= \frac{1}{2}\log\left(\frac{1}{1+1/s^2}\right)\Bigg|_s^\infty = \frac{1}{2}\left[0 - \log\left(\frac{1}{1+1/s^2}\right)\right] \\
&= \frac{1}{2}\left[-\log\left(\frac{s^2}{s^2+1}\right)\right] = -\frac{1}{2}\log\left(\frac{s^2}{s^2+1}\right)
\end{aligned}$$

Again

$$L\left[\frac{1-\cos t}{t^2}\right] = L\left[\frac{(\frac{1-\cos t}{t})}{t}\right]$$
$$= \int_s^\infty L\left[\frac{1-\cos t}{t}\right] \mathrm{d}s$$
$$= \int_s^\infty -\frac{1}{2}\log\left(\frac{s^2}{s^2+1}\right) \mathrm{d}s$$
$$= -\frac{1}{2}\int_s^\infty \log\left(\frac{s^2}{s^2+1}\right) 1\,\mathrm{d}s$$

Integrating by parts

$$= -\frac{1}{2}\left\{\left[\log\left(\frac{s^2}{s^2+1}\right)s\right]_s^\infty - \int_s^\infty \frac{1}{(\frac{s^2}{s^2+1})}\right.$$
$$\left.\frac{[(s^2+1)2s - s^2\cdot 2s]}{(s^2+1)^2} \times s\,\mathrm{d}s\right\}$$
$$= -\frac{1}{2}\left\{\left[s\log\left(\frac{1}{1+\frac{1}{s^2}}\right)\right]_s^\infty - \int_s^\infty \frac{s^2+1}{s^2} \times \frac{2s^2}{(s^2+1)^2}\mathrm{d}s\right\}$$
$$= -\frac{1}{2}\left\{\left[0 - s\cdot\log\left(\frac{1}{1+\frac{1}{s^2}}\right)\right] - \int_s^\infty \frac{2}{s^2+1}\mathrm{d}s\right\}$$
$$= -\frac{1}{2}\left\{-s\log\left(\frac{s^2}{s^2+1}\right) - 2(\tan^{-1}(s))_s^\infty\right\}$$
$$= \frac{s}{2}\log\left(\frac{s^2}{s^2+1}\right) + \frac{\pi}{2} - \tan^{-1}s$$
$$= \frac{s}{2}\log\left(\frac{s^2}{s^2+1}\right) + \frac{\pi}{2} - \tan^{-1}s$$
$$= \frac{s}{2}\log\left(\frac{s^2}{s^2+1}\right) + \cot^{-1}s$$

EXERCISE

Find the Laplace transform of the following

(1) $\cosh at \sin at$	(2) $\mathrm{e}^{-t}\cos^2 t$
(3) $t^3\mathrm{e}^{2t}$	(4) $\mathrm{e}^{-t}(2\cos 5t - 3\sin 5t)$
(5) $2\times e^{-t}\cos^2\frac{1}{2}t$	(6) $\mathrm{e}^{4t}\sin 2t\cos t$
(7) $\cosh 3t\cos 2t$	(8) $\mathrm{e}^{-t}(\sin 2t - 2t\cos 2t)$
(9) $\mathrm{e}^{-t}\cos^2 t\sin 3t$	(10) $t^2\mathrm{e}^{-2t}$

(11) $\dfrac{\sin^2 t}{t}$ (12) $\dfrac{\sinh t}{t}$

(13) $t^2 e^{-2t}\cos t$ (14) $\dfrac{e^{-t}\sin t}{t}$

(15) $\dfrac{e^{at}-\cos bt}{t}$

(16) $\dfrac{\cos 2t-\cos 3t}{t}$ (17) $t^2 e^{-3t}\sin 2t$

ANSWERS

(1) $\dfrac{a(s^2+2a^2)}{s^4+4a^4}$ (2) $\dfrac{1}{2s+2}+\dfrac{s+1}{2s^2+4s+10}$

(3) $\dfrac{6}{(s-2)^4}$ (4) $\dfrac{2s-13}{s^2+2s+26}$

(5) $\dfrac{1}{(s+1)(s^2+2s+2)}$ (6) $\dfrac{1}{2}\left[\dfrac{3}{(s-4)^2+9}+\dfrac{1}{(s-4)^2+1}\right]$

(7) $\dfrac{2s(s^2-5)}{s^4-10s^2+169}$ (8) $\dfrac{16}{(s^2+2s+5)^2}$

(9) $\dfrac{1}{4}\left[\dfrac{6}{(s+1)^2+9}+\dfrac{5}{(s+1)^2+25}+\dfrac{1}{(s+1)^2+1}\right]$

(10) $\dfrac{2}{(s+2)^4}$ (11) $\dfrac{1}{4}\log\left(\dfrac{s^2+4}{s^2}\right)$

(12) $\dfrac{1}{2}\log\left(\dfrac{s-1}{s+1}\right)$ (13) $\dfrac{2(s^3+10s^2+25s+22)}{(s^2+4s+5)^3}$

(14) $\cot^{-1}(s+1)$ (15) $\dfrac{1}{2}\log\left(\dfrac{s^2-b^2}{s-a}\right)$

(16) $\dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+4}\right)$ (17) $\dfrac{2(s^3+6s^2+9s+2}{(s^2+4s+5)^3}$

9 INVERSE LAPLACE TRANSFORMS

If $\mathscr{L}[f(t)]=F(s)$, then $\mathscr{L}^{-1}[F(s)]=f(t)$, where L^-1 is called the inverse Laplace transform operator. From the application point of view, the inverse Laplace transform is very useful.

Important formulae

(1) $\mathscr{L}^{-1}\left[\dfrac{1}{s}\right]=1$

(2) $\mathscr{L}^{-1}\left[\dfrac{1}{s^n}\right]=\dfrac{t^{n-1}}{(n-1)!}$

(3) $\mathscr{L}^{-1}\left[\dfrac{1}{s-a}\right]=e^{at}$

(4) $\mathscr{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

(5) $\mathscr{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

(6) $\mathscr{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

(7) $\mathscr{L}^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{sinhat}{a}$

(8) $\mathscr{L}^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

(9) Shifting property:
$\mathscr{L}^{-1}[F(s-a)] = e^{at}\mathscr{L}^{-1}[F(s)]$

(10) Multiplication by t
$tf(t) = \mathscr{L}^{-1}\left\{-\frac{d}{ds}\mathscr{L}(f(t))\right\}$

(11) Division by t
$\frac{f(t)}{t} = \mathscr{L}^{-1}\left\{\int_s^\infty L(f(t))ds\right\}$

EXAMPLE 9

Find the inverse transform of the follwing

(i)	$\frac{s^2+2s+5}{s^3}$	(ii)	$\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$
(iii)	$\frac{4s+5}{(s+1)^2(s-2)}$	(iv)	$\frac{2s+5}{s^2+4s-5}$
(v)	$\frac{2s+5}{s^2+4s+13}$	(vi)	$\frac{5s+3}{(s-1)(s^2+2s+5)}$
(vii)	$\frac{s^2+6}{(s^2+1)(s^2+4)}$	(viii)	$\frac{s}{s^4+4a^4}$
(ix)	$\frac{s}{s^4+s^2+1}$	(x)	$\log\left(\frac{s+1}{s-1}\right)$
(xi)	$\log\left(\frac{1+s}{s}\right)$	(xii)	$\cot^{-1}\left(\frac{s}{2}\right)$

Solution

(i) $\dfrac{s^2+2s+5}{s^3} = \dfrac{s^2}{s^3}+\dfrac{2s}{s^3}+\dfrac{5}{s^3} = \dfrac{1}{s}+\dfrac{2}{s^2}+\dfrac{5}{s^3}$

$$\begin{aligned}
\therefore \mathscr{L}^{-1}\left[\frac{s^2+2s+5}{s^3}\right] &= \mathscr{L}^{-1}\left(\frac{1}{s}\right)+2\mathscr{L}^{-1}\left(\frac{1}{s^2}\right)+5\mathscr{L}^{-1}\left(\frac{1}{s^3}\right)\\
&= 1+2\times\frac{t}{1!}+5\times\frac{t^2}{2!}\\
&\quad \left(\because \mathscr{L}^{-1}(t^n)=\frac{t^{n-1}}{(n-1)!}\right)\\
&= 1+2t+\frac{5}{2}t^2
\end{aligned}$$

(ii) Let $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6} = \dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$
$= \dfrac{A}{s-1}+\dfrac{B}{s-2}+\dfrac{C}{s-3}$

Then (by resolving into partial fraction)

$$\begin{aligned}
\text{Putting } s=1,\quad A &= \frac{2\cdot 1^2-6\cdot 1+5}{(1-2)(1-3)} = 1/2\\
\text{Putting } s=2,\quad B &= \frac{2\cdot 2^2-6\cdot 2+5}{(2-1)(2-3)} = -1\\
\text{Putting } s=3,\quad c &= \frac{2\cdot 3^2-6\cdot 3+5}{(3-1)(3-2)} = 5/2.\\
\therefore \frac{2s^2-6s+5}{s^3-6s^2+11s-6} &= \frac{1/2}{s-1}+\frac{-1}{s-2}+\frac{5/2}{s-3}\\
\therefore \mathscr{L}^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right] &= \frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{s-1}\right]-\mathscr{L}^{-1}\left[\frac{1}{s-2}\right]\\
&\quad +\frac{5}{2}\mathscr{L}^{-1}\left[\frac{1}{s-3}\right]\\
&= \frac{1}{2}e^t-e^{2t}+\frac{5}{2}e^{3t}
\end{aligned}$$

(iii) Let $\dfrac{4s+5}{(s+1)^2(s-2)} = \dfrac{A}{s+1}+\dfrac{B}{(s+1)^2}+\dfrac{C}{s-2}$

Multiplying both sides by $(s+1)^2(s-2)$,

$$4s+5 = A(s+1)(s-2)+B(s-2)+c(s+1)^2$$

$$\text{Put } s=2,\qquad 13=9C \Rightarrow C=13/9$$

$$\text{Put } s=-1,\qquad 1=-3B \Rightarrow B=-\frac{1}{3}$$

Equating the coefficients of s^2,

$$0 = A+C \Rightarrow A=-C=-13/9.$$

$$\therefore \frac{4s+5}{(s+1)^2(s-2)} = \frac{-13/9}{s+1} + \frac{-1/3}{(s+1)^2} + \frac{13/9}{(s-2)}$$

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{4s+5}{(s+1)^2(s-2)}\right] &= -\frac{13}{9}\mathscr{L}^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{3}\mathscr{L}^{-1}\left[\frac{1}{(s+1)^2}\right] \\ &\quad + \frac{13}{9}\mathscr{L}^{-1}\left[\frac{1}{s-2}\right] \\ &= -\frac{13}{9}e^{-t} - \frac{1}{3}te^{-t} + \frac{13}{9}e^{2t}.\end{aligned}$$

(iv) Let $\frac{2s+5}{s^2+4s-5} = \frac{2s+5}{(s-1)(s+5)} = \frac{A}{s-1} + \frac{B}{s+5}$

Then

$$\begin{aligned} A &= \frac{2\cdot 1+5}{1+5} = 7/6 \\ B &= \frac{2\cdot(-5)+5}{(-5-1)} = \frac{-5}{-6} = 5/6 \\ \therefore \frac{2s+5}{s^2+4s-5} &= \frac{7/6}{s-1} + \frac{5/6}{s+5} \\ \text{i.e. } \mathscr{L}^{-1}\left[\frac{2s+5}{s^2+4s-5}\right] &= \frac{7}{6}\mathscr{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{5}{6}\mathscr{L}^{-1}\left[\frac{1}{s+5}\right] \\ &= \frac{7}{6}e^{t} + \frac{5}{6}e^{-5t}\end{aligned}$$

(v) Let $F(s) = \frac{2s+5}{s^2+4s+13}$ where the denominator is not factorizable

$$\begin{aligned} F(s) &= \frac{2s+5}{(s+2)^2-4+13} = \frac{2(s+2)+1}{(s+2)^2+9} \\ &= \frac{2(s+2)}{(s+2)^2+9} + \frac{1}{(s+2)^2+9} \\ \therefore \mathscr{L}^{-1}[F(s)] &= 2\mathscr{L}^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] + \mathscr{L}^{-1}\left[\frac{1}{(s+2)^2+3^2}\right] \\ &= 2e^{-2t}\mathscr{L}^{-1}\left(\frac{s}{s^2+3^2}\right) + e^{-2t}\mathscr{L}^{-1}\left[\frac{1}{s^2+3^2}\right] \\ &\quad (\because \mathscr{L}^{-1}[F(s+a)] = e^{-at}\mathscr{L}^{-1}[F(s)]) \\ &= 2e^{-2t}\cos 3t + e^{-2t}\frac{\sin 3t}{3}\end{aligned}$$

(vi) Let $\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$

Multiplying both sides by $(s-1)(s^2+2s+5)$,

$$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$$

Put $s=1$, $\quad 8 = A\times 8 \Rightarrow A = 1$

Put $s=0$, $\quad 3 = 5A - C \Rightarrow C = 5A-3 = 5-3 = 2$

Equating the coefficients of s^2,

$$\begin{aligned}
0 &= A+B \Rightarrow B=-A=-1\\
\therefore \frac{5s+3}{(s-1)(s^2+2s+5)} &= \frac{1}{s-1}+\frac{-s+2}{s^2+2s+5}\\
\mathscr{L}^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] &= \mathscr{L}^{-1}\left[\frac{1}{s-1}\right]+\mathscr{L}^{-1}\left[\frac{-s+2}{s^2+2s+5}\right]\\
&= \mathscr{L}^{-1}\left[\frac{1}{s-1}\right]+\mathscr{L}^{-1}\left[\frac{-(s+1)+3}{(s+1)^2-1+5}\right]\\
&= \mathscr{L}^{-1}\left[\frac{1}{s-1}\right]+\mathscr{L}^{-1}\left[\frac{-(s+1)}{(s+1)^2+2^2}\right]\\
&\quad +3\mathscr{L}^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]\\
&= e^{+t}-e^{-t}\mathscr{L}^{-1}\left[\frac{s}{s^2+2^2}\right]+3e^{-t}\mathscr{L}^{-1}\left(\frac{1}{s^2+2^2}\right)\\
&\quad (\because \mathscr{L}^{-1}[F(s+a)]=e^{-t}\mathscr{L}^{-1}[F(s)])\\
&= e^{t}-e^{-t}\cos 2t+3e^{-t}\cdot\frac{\sin 2t}{2}
\end{aligned}$$

(vii) Let $F(s)=\dfrac{s^2+6}{(s^2+1)(s^2+4)}$.

Since $F(s)$ involves only even powers of s, we put $s^2=u$

$$F(s)=\frac{u+6}{(u+1)(u+4)}=\frac{A}{u+1}+\frac{B}{u+4}$$

where

$$\begin{aligned}
A &= \frac{-1+6}{-1+4}=\frac{5}{3} \qquad \text{and} \qquad B=\frac{-4+6}{-4+1}=\frac{2}{-3}\\
\therefore F(s) &= \frac{5/3}{u+1}+\frac{-2/3}{u+4}=\frac{5}{3}\cdot\frac{1}{s^2+1}-\frac{2}{3}\cdot\frac{1}{s^2+4}\\
\therefore \mathscr{L}^{-1}[F(s)] &= \frac{5}{3}\mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right]-\frac{2}{3}\mathscr{L}^{-1}\left[\frac{1}{s^2+4}\right]\\
&= \frac{5}{3}\cdot\frac{\sin t}{1}-\frac{2}{3}\cdot\frac{\sin 2t}{2}\\
&= \frac{5}{3}\sin t-\frac{1}{3}\sin 2t
\end{aligned}$$

(viii) Since $s^4+4a^4=(s^2)^2+(2a^2)^2$

$$\begin{aligned}
&= (s^2+2a^2)^2-2s^2\cdot 2a^2\\
&\quad (\because A^2+B^2=(A+B)^2-2AB)\\
&= (s^2+2a^2)^2-(2as)^2\\
&= (s^2+2a^2+2as)(s^2+2a^2-2as)
\end{aligned}$$

Let

$$\begin{aligned}
\frac{s}{s^4+4a^4} &= \frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)} \\
&= \frac{As+B}{s^2+2as+2a^2}+\frac{Cs+D}{s^2-2as+2a^2} \\
&= \frac{-1/4a}{s^2+2as+2a^2}+\frac{1/4a}{s^2-2as+2a^2} \\
&= -\frac{1}{4a}\times\frac{1}{(s+a)^2-a^2+2a^2}+\frac{1}{4a} \\
&\times\frac{1}{(s-a)^2-a^2+2a^2} \\
&= -\frac{1}{4a}\times\frac{1}{(s+a)^2+a^2}+\frac{1}{4a}\times\frac{1}{(s-a)^2+a^2} \\
\mathscr{L}^{-1}\left[\frac{s}{s^4+4a^4}\right] &= -\frac{1}{4a}\times \mathrm{e}^{-at}\times\mathscr{L}^{-1}\left(\frac{1}{s^2+a^2}\right)+\frac{1}{4a} \\
&\times \mathrm{e}^{at}\mathscr{L}^{-1}\left(\frac{1}{s^2+a^2}\right) \\
&= -\frac{1}{4a}\mathrm{e}^{-at}\frac{\sin at}{a}+\frac{1}{4a}\mathrm{e}^{at}\frac{\sin at}{a} \\
&= \frac{\sin at}{4a^2}(\mathrm{e}^{at}-\mathrm{e}^{-at}) = \frac{\sin at}{2a^2}\times\sinh at
\end{aligned}$$

(ix) Since $s^4+s^2+1 = s^4+2s^2+1-s^2$

$$\begin{aligned}
&= (s^2+1)^2-s^2 \\
&= (s^2+1+s)(s^2+1-s)
\end{aligned}$$

Let

$$\frac{s}{s^4+s^2+1} = \frac{s}{(s^2+s+1)(s^2-s+1)} = \frac{As+B}{s^2+s+1}+\frac{Cs+D}{s^2-s+1}$$

Multiplying both sides by s^4+s^2+1,

$$s = (As+B)(s^2-s+1)+(Cs+D)(s^2+s+1)$$

Equating the coefficients of s^3,

$$O = A+C \tag{1}$$

Equating the coefficients of s^2,

$$O = B-A+C+D \tag{2}$$

Equating the coefficients of s,

$$1 = A-B+C+D \tag{3}$$

Equating the coefficient terms,

$$O = B + D \tag{4}$$

By (4), $B + D = O$ substituting in (2),

(2) becomes

$$O = -A + C \Rightarrow A = C$$

Then (1) becomes,

$$O = A + A \Rightarrow 2A = O \Rightarrow A = O$$

and hence

$$C = O.$$

From (3),

$$1 = D - B$$

and by (4),

$$O = D + B$$

Adding

$$1 = 2D \Rightarrow D = \frac{1}{2} \qquad \text{and} \qquad B = -D = -\frac{1}{2}$$

$$\begin{aligned}
\therefore \frac{s}{s^4+s^2+1} &= \frac{-1/2}{s^2+s+1} + \frac{1/2}{s^2-s+1} \\
&= \frac{-1/2}{(s+\frac{1}{2})^2 - \frac{1}{4} + 1} + \frac{1/2}{(s-\frac{1}{2})^2 - \frac{1}{4} + 1} \\
&= \frac{-1/2}{(s+\frac{1}{2})^2 + \frac{3}{4}} + \frac{1/2}{(s-\frac{1}{2})^2 + \frac{3}{4}} \\
\therefore \mathscr{L}^{-1}\left[\frac{s}{s^4+s^2+1}\right] &= -\frac{1}{2} \times e^{-t/2} \mathscr{L}^{-1}\left[\frac{1}{s^2 + \left(\sqrt{3}/2\right)^2}\right] \\
&\quad + \frac{1}{2} e^{t/2} \mathscr{L}^{-1}\left(\frac{1}{s^2 + (\sqrt{3}/2)^2}\right) \\
&= -\frac{1}{2} e^{-t/2} \frac{\sin \sqrt{3}t/2}{\sqrt{3}/2} + \frac{1}{2} \times e^{t/2} \times \frac{\sin \sqrt{3}t/2}{\sqrt{3}/2} \\
&= \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \left(\frac{e^{t/2} - e^{-t/2}}{2}\right) \\
&= \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \sinh \frac{t}{2}
\end{aligned}$$

(x) Let $\mathscr{L}[f(t)] = F(s) = \log\left(\frac{s+1}{s-1}\right) = \log(s+1) - \log(s-1)$

Then

$$\begin{aligned}
\mathscr{L}[tf(t)] &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{L}[f(t)] \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}[\log(s+1) - \log(s-1)] \\
&= -\left[\frac{1}{s+1} - \frac{1}{s-1}\right] \\
&= \frac{1}{s-1} - \frac{1}{s+1} \\
\therefore tf(t) &= \mathscr{L}^{-1}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] = \mathrm{e}^{t} - \mathrm{e}^{-t} \\
\therefore f(t) &= \frac{e^{t} - e^{-t}}{t} \qquad \text{or} \qquad \mathscr{L}^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = \frac{e^{t} - e^{-t}}{t}
\end{aligned}$$

(xi) Let $\mathscr{L}[f(t)] = F(s) = \log\left(\frac{s+1}{s}\right) = \log(s+1) - \log s$

Then

$$f(t) = \mathscr{L}^{-1}\left[\log\left(\frac{s+1}{s}\right)\right]$$

$$\begin{aligned}
\text{But} \quad \mathscr{L}[tf(t)] &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{L}[f(t)] \\
&= -\frac{\mathrm{d}}{\mathrm{d}s}[\log(s+1) - \log(s) \\
&= -\left[\frac{1}{s+1} - \frac{1}{s}\right] \\
&= \frac{1}{s} - \frac{1}{s+1} \\
\therefore tf(t) &= \mathscr{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = 1 - \mathrm{e}^{-t} \\
\therefore f(t) &= \frac{1 - \mathrm{e}^{-t}}{t} \\
\text{or} \qquad \mathscr{L}^{-1}\left[\log\left(\frac{s+1}{s}\right)\right] &= \frac{1 - \mathrm{e}^{-t}}{t}
\end{aligned}$$

(xii) Let $\mathscr{L}[f(t)] = F(s) = \cot^{-1}(s/2)$

Then

$$f(t) = \mathscr{L}^{-1}[\cot^{-1}(s/2)]$$

But

$$\begin{aligned}
\mathscr{L}[tf(t)] &= -\frac{d}{ds}\mathscr{L}[f(t)] \\
&= -\frac{d}{ds}[\cot^{-1}(s/2)] \\
&= -\left[\frac{-2}{s^2+2^2}\right] = \frac{2}{s^2+2^2} \\
\therefore tf(t) &= \mathscr{L}^{-1}\left[\frac{2}{s^2+2^2}\right] = \sin 2t \\
\therefore f(t) &= \frac{\sin 2t}{t}
\end{aligned}$$

EXERCISE 4.3

Find the inverse transform of the following

(i)	$\frac{3(s^2-1)^2}{2s^5}$	(ii)	$\frac{s+1}{s^2+s+1}$
(iii)	$\frac{2s^2-4}{(s+1)(s^2-5s+6)}$	(iv)	$\frac{2s^2-1}{(s^2+1)(s^2+4)}$
(v)	$\frac{21s-33}{(s+1)(s-2)^3}$	(vi)	$\frac{s^3}{s^4-a^4}$
(vii)	$\frac{s+8}{s^2+4s+5}$	(viii)	$\frac{2s+1}{(s+2)^2(s-1)^2}$
(ix)	$\log\left(1+\frac{1}{s^2}\right)$	(x)	$\frac{3s-8}{s^2-4s+20}$
(xi)	$\frac{s+1}{(s^2+2s+2)^2}$	(xii)	$\frac{s-4}{4(s-3)^2+16}$
(xiii)	$\frac{1}{2(s-1)^2+32}$	(xiv)	$\tan^{-1}\left(\frac{1}{s}\right)$
(xv)	$\log\left(\frac{s^2-1}{s^2}\right)$	(xvi)	$\cot^{-1}(1+s)$
(xvii)	$\frac{1}{2}\log\left(\frac{s^2+b^2}{(s-a)^2}\right)$	(xviii)	$\frac{1}{(s+1)(s^2+2s+2)}$
(xix)	$\frac{s}{(s^2-1)^2}$	(xx)	$\frac{s+3}{(s^2+6s+13)^2}$
(xxi)	$\frac{s^2+s}{(s^2+1)(s^2+2s+2)}$	(xxii)	$\log\left(\frac{s+a}{s+b}\right)$
(xxiii)	$\frac{11s^2-2s+5}{2s^3-3s^2-3s+2}$		

ANSWERS

(i)	$\frac{3}{2} - \frac{3}{2}t^2 + \frac{t^4}{16}$	(ii)	$\frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \left(\sqrt{2} \cos \frac{\sqrt{3}}{2} t + \sin \frac{\sqrt{3}}{2} t \right)$
(iii)	$\frac{7}{2}e^{3t} - \frac{1}{6}e^{-t} - \frac{4}{3}e^{2t}$	(iv)	$\frac{3}{2}\sin 2t - \sin t$
(v)	$2e^{-t} - 2e^{2t} + 6te^{2t} + \frac{3}{2}t^2e^{2t}$	(vi)	$\frac{\cosh at + \cos at}{2}$
(vii)	$e^{-2t}(\cos t + 6\sin t)$	(viii)	$\frac{t(e^t - e^{-2t})}{3}$
(ix)	$\frac{2(1 - \cos t)}{t}$	(x)	$e^{2t}(3\cos 4t - \frac{1}{2}\sin 4t)$
(xi)	$\frac{1}{2}te^{-t}\sin t$	(xii)	$\frac{1}{4}e^{3t}\cos 2t - \frac{1}{8}e^{3t}\sin 2t$
(xiii)	$\frac{e^t}{8}\sin 4t$	(xiv)	$\frac{\sin t}{t}$
(xv)	$\frac{2}{t}(1 - \cosh t)$	(xvi)	$\frac{e^{-t}\sin t}{t}$
(xvii)	$\frac{e^{-at} - \cos bt}{t}$	(xviii)	$e^{-t}(1 - \cos t)$
(xix)	$\frac{1}{2}t\sinh t$	(xx)	$3e^{-3t}t\sin t$
(xxi)	$\frac{1}{5}(1 + e^{-t})\sin t + \frac{3}{5}(1 - e^{-t})\cos t$		
(xxii)	$\frac{e^{-bt} - e^{-at}}{t}$	(xxiii)	$2e^{-t} + 5e^{2t} - \frac{3}{2}e^{t/2}$

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